Ensuring Efficient Hedging of Barrier Options

Uwe Wystup
Commerzbank Treasury and Financial Products
Mainzer Landstr. 153
60261 Frankfurt am Main
GERMANY
phone: +49 - 69 - 136 - 41067
uwe@mathfinance.de
http://www.mathfinance.de

February 5, 2002

Contents

1 Introduction

Theoretical Black-Scholes prices can differ from market prices substantially if the derivative has

1. kinks
2. jumps
3. barries

because of the hedging difficulties created by their large delta and gamma values. We consider barrier options as a typical example of first-generation exotics. One problem for practitioners is to quantify the price of difficult hedging systematically. Adjusting the volatility is inconsistent since option values are not guaranteed to be montone in the volatility.

1.1 Digital options

1.1.1 payoff

$$\begin{aligned} v(T) &= I_{\{\phi S_T \geq \phi K\}} \text{ cash-or-nothing,} && (1)\\ w(T) &= S_T I_{\{\phi S_T \geq \phi K\}} \text{ asset-or-nothing.} && (2) \end{aligned}$$

In the cash-or-nothing case the payment of the fixed amount is in domestic currency, whereas in the asset-or-nothing case the payment is in foreign currency. We use the abbreviations

1.1.2 valuation in the Black-Scholes model

$$\begin{aligned} F &\triangleq \mathbb{E}[S_T | S_t = x] = xe^{(r_d - r_f)\tau} \text{ (forward price of the underlying) ,} && (3)\\ d_\pm &\triangleq \frac{\ln \frac{x}{K} + \sigma\theta_\pm \tau}{\sigma\sqrt{\tau}} = \frac{\ln \frac{F}{K} \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}, && (4)\\ \tilde{d}_\pm &\triangleq \frac{\ln \frac{x}{K} - \sigma\theta_\pm \tau}{\sigma\sqrt{\tau}}, && (5) \end{aligned}$$

and obtain for the value functions

$$\begin{aligned} v(x, K, T, t, \sigma, r_d, r_f, \phi) &= e^{-r_d\tau}\mathcal{N}(\phi d_-), && (6)\\ w(x, K, T, t, \sigma, r_d, r_f, \phi) &= xe^{-r_f\tau}\mathcal{N}(\phi d_+). && (7) \end{aligned}$$

1.1.3 Greeks

(Spot) Delta

$$\begin{aligned} \frac{\partial v}{\partial x} &= \phi e^{-r_d\tau}\frac{n(d_-)}{x\sigma\sqrt{\tau}} && (8)\\ \frac{\partial w}{\partial x} &= \phi e^{-r_f\tau}\frac{n(d_+)}{\sigma\sqrt{\tau}} + e^{-r_f\tau}\mathcal{N}(\phi d_+) && (9) \end{aligned}$$

Gamma

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= -\phi e^{-r_d\tau}\frac{n(d_-)d_+}{x^2\sigma^2\tau} && (10)\\ \frac{\partial^2 w}{\partial x^2} &= -\phi e^{-r_f\tau}\frac{n(d_+)d_-}{x\sigma^2\tau} && (11) \end{aligned}$$

Theta

$$\begin{aligned}\frac{\partial v}{\partial t} &= e^{-r_d\tau}\left(r_d\mathcal{N}(\phi d_-)+\frac{\phi n(d_-)\tilde{d}_-}{2\tau}\right) &(12)\\ \frac{\partial w}{\partial t} &= xe^{-r_f\tau}\left(r_f\mathcal{N}(\phi d_+)+\frac{\phi n(d_+)\tilde{d}_+}{2\tau}\right) &(13)\end{aligned}$$

Vega

$$\begin{aligned}\frac{\partial v}{\partial \sigma} &= -\phi e^{-r_d\tau}n(d_-)\frac{d_+}{\sigma} &(14)\\ \frac{\partial w}{\partial \sigma} &= -\phi xe^{-r_f\tau}n(d_+)\frac{d_-}{\sigma} &(15)\end{aligned}$$

Volga

$$\frac{\partial^2 v}{\partial \sigma^2} = -\phi e^{-r_d\tau}\frac{n(d_-)}{\sigma^2}(d_-d_+^2-d_--d_+) \qquad (16)$$

$$\frac{\partial^2 w}{\partial \sigma^2} = -\phi xe^{-r_f\tau}\frac{n(d_+)}{\sigma^2}(d_+d_-^2-d_+-d_-) \qquad (17)$$

Rho

$$\begin{aligned}\frac{\partial v}{\partial r_d} &= e^{-r_d\tau}\left(-\tau\mathcal{N}(\phi d_-)+\frac{\phi\sqrt{\tau}n(d_-)}{\sigma}\right) &(18)\\ \frac{\partial v}{\partial r_f} &= e^{-r_d\tau}\left(-\frac{\phi\sqrt{\tau}n(d_-)}{\sigma}\right) &(19)\\ \frac{\partial w}{\partial r_d} &= xe^{-r_f\tau}\left(\frac{\phi\sqrt{\tau}n(d_+)}{\sigma}\right) &(20)\\ \frac{\partial w}{\partial r_f} &= -xe^{-r_f\tau}\left(\tau\mathcal{N}(\phi d_+)+\frac{\phi\sqrt{\tau}n(d_+)}{\sigma}\right) &(21)\end{aligned}$$

Dual Delta

$$\begin{aligned}\frac{\partial v}{\partial K} &= -e^{-r_d\tau}\frac{\phi n(d_-)}{K\sigma\sqrt{\tau}} &(22)\\ \frac{\partial w}{\partial K} &= -e^{-r_d\tau}\frac{\phi n(d_-)}{\sigma\sqrt{\tau}} &(23)\end{aligned}$$

Dual Gamma

$$\begin{aligned}\frac{\partial^2 v}{\partial K^2} &= \phi e^{-r_d\tau}\frac{n(d_-)}{K^2\sigma^2\tau}(\sigma\sqrt{\tau}-d_-) &(24)\\ \frac{\partial^2 w}{\partial K^2} &= -\phi e^{-r_d\tau}\frac{n(d_-)d_-}{K\sigma^2\tau} &(25)\end{aligned}$$

Dual Theta

$$\frac{\partial v}{\partial T} = -v_t \qquad (26)$$

1.1.4 Foreign-domestic symmetry

One can directly verify the relationship

$$\frac{1}{x}v(x,K,T,t,\sigma,r_d,r_f,\phi) = w(\frac{1}{x},\frac{1}{K},T,t,\sigma,r_f,r_d,-\phi). \tag{27}$$

The reason is that the value of an option can be computed both in a domestic as well as in a foreign scenario. We consider the example of S_t modeling the exchange rate of EUR/USD. In New York, the cash-or-nothing digital call option costs $v(x,K,T,t,\sigma,r_{usd},r_{eur},1)$ USD and hence $v(x,K,T,t,\sigma,r_{usd},r_{eur},1)/x$ EUR. If it ends in the money, the holder receives 1 USD. For a Frankfurt-based holder of the same option, receiving one USD means receiving asset-or-nothing, where he uses reciprocal values for spot and strike and for him domestic currency is the one that's foreign to the New Yorker and vice versa. Since S_t and $\frac{1}{S_t}$ have the same volatility, the New York value and the Frankfurt value must agree, which leads to (27).

1.1.5 Relationship between cash, asset and vanilla

The simple equation of payoffs

$$\phi(w(T) - Kv(T)) = [\phi(S_T - K)]^+ \tag{28}$$

leads to the formula

$$\begin{aligned} & \text{vanilla}(x,K,T,t,\sigma,r_d,r_f,\phi) \\ = \quad & \phi[w(x,K,T,t,\sigma,r_d,r_f,\phi) - Kv(x,K,T,t,\sigma,r_d,r_f,\phi)]. \end{aligned} \tag{29}$$

1.1.6 Static hedge using vertical spreads

The mathematical derivative of the positive part function

$$I_{\{\phi S_t \geq \phi K\}} = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\left[(\phi(S_T - (K - \phi\epsilon)))^+ - (\phi(S_T - (K + \phi\epsilon)))^+\right] \tag{30}$$

leads to an approximate static hedge (and hence price)

$$\begin{aligned} & v(x,K,T,t,\sigma,r_d,r_f,\phi) \approx \\ & \frac{1}{2\epsilon}\left[\text{vanilla}(x,K-\phi\epsilon,T,t,\sigma,r_d,r_f,\phi) - \text{vanilla}(x,K+\phi\epsilon,T,t,\sigma,r_d,r_f,\phi)\right] \end{aligned} \tag{31}$$

for small $\epsilon > 0$. In practice, arbitrarily small ϵ corresponds to arbitrarily large nominal amounts of the vanilla options and can thus not be chosen arbitrarily small. Furthermore, there will be different volatilities for the bid and ask price of the vanilla options, which lead to a more realistic pricing for digital options using this approximation.

Greeks in the static hedge Static hedges normally perform well hedging the actual model variable risk like delta, gamma and theta. In this static hedge even the model *parameter* uncertainty vega is hedged. The hedge vega is given by

$$\sqrt{\tau} x e^{-r_f\tau} \frac{n(d_+^{K-\phi\epsilon}) - n(d_+^{K+\phi\epsilon})}{2\epsilon}, \tag{32}$$

$$d_\pm^K \triangleq \frac{\ln\frac{F}{K} \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}. \tag{33}$$

Replacing the difference quotient by its derivative at K we obtain

$$\begin{aligned} & \sqrt{\tau} x e^{-r_f\tau} \frac{n(d_+^{K-\phi\epsilon}) - n(d_+^{K+\phi\epsilon})}{2\epsilon} && (34)\\ \approx\ & \phi\sqrt{\tau} x e^{-r_f\tau} \cdot n(d_+) d_+ \frac{-1}{K\sigma\sqrt{\tau}} && (35)\\ =\ & -\phi e^{-r_d\tau} n(d_-) \frac{d_+}{\sigma}, && (36) \end{aligned}$$

which is the vega of the digital option.

1.2 One-touch options

1.2.1 payoff

$$R I_{\{\tau_B \le T\}}, \tag{37}$$

$$\tau_B \triangleq \inf\{t \ge 0 : \eta S_t \le \eta B\}. \tag{38}$$

This type of option pays a domestic cash amount R if a barrier B is hit any time before expiry. We use the binary variable η to describe whether B is a lower barrier ($\eta = 1$) or an upper barrier ($\eta = -1$). The stopping time τ_B is called the *first hitting time*.

1.2.2 other names

- rebate portion of a knock-out barrier option
- American cash-or-nothing digital option
- one-touch-digital
- hit options

1.2.3 rebates for knock-in options

The modified payoff $RI_{\{\tau_B \ge T\}}$ describes a rebate which is paid if a knock-in-option has not knocked in by the time it expires and can be valued similarly simply by exploiting the identity

$$R I_{\{\tau_B \le T\}} + R I_{\{\tau_B \ge T\}} = R. \tag{39}$$

1.2.4 payment date

We will further distinguish whether the rebate is paid at hit ($\omega = 0$) or at end ($\omega = 1$) and use the abbreviations

$$\vartheta_- \triangleq \sqrt{\theta_-^2 + 2(1-\omega) r_d}, \tag{40}$$

$$e_\pm \triangleq \frac{\pm \log\frac{x}{B} - \sigma\vartheta_-\tau}{\sigma\sqrt{\tau}}. \tag{41}$$

1.2.5 Pricing

The value of the one-touch option turns out to be

$$v(t,x) = R e^{-\omega r_d \tau} \left[\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} \mathcal{N}(-\eta e_+) + \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} \mathcal{N}(\eta e_-) \right]. \tag{42}$$

Note that $\vartheta_- = |\theta_-|$ for rebates paid at end ($\omega = 1$).

1.2.6 Greeks

Delta

$$\begin{aligned} v_x(t,x) = -\frac{Re^{-\omega r_d\tau}}{\sigma x}\Bigg\{&\left(\frac{B}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}\left[(\theta_-+\vartheta_-)\mathcal{N}(-\eta e_+)+\frac{\eta}{\sqrt{\tau}}n(e_+)\right] \\ &+\left(\frac{B}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}\left[(\theta_--\vartheta_-)\mathcal{N}(\eta e_-)+\frac{\eta}{\sqrt{\tau}}n(e_-)\right]\Bigg\} \end{aligned} \tag{43}$$

Theta

$$\begin{aligned} v_t(t,x) &= \omega r_d v(t,x)+\frac{\eta R e^{-\omega r_d\tau}}{2\tau}\left[\left(\frac{B}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}n(e_+)e_- - \left(\frac{B}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}n(e_-)e_+\right] \\ &= \omega r_d v(t,x)+\frac{\eta R e^{-\omega r_d\tau}}{\sigma\tau^{(3/2)}}\left(\frac{B}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}n(e_+)\log\left(\frac{B}{x}\right). \end{aligned} \tag{44}$$

The computation exploits the identities (61), (62) and (63) derived below.

Gamma Gamma can be obtained using $v_{xx}=\frac{2}{\sigma^2x^2}[r_dv-v_t-(r_d-r_f)xv_x]$ and turns out to be

$$\begin{aligned} v_{xx}(t,x) = &\frac{2Re^{-\omega r_d\tau}}{\sigma^2x^2}\cdot \\ &\Bigg\{\left(\frac{B}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}\mathcal{N}(-\eta e_+)\left[r_d(1-\omega)+(r_d-r_f)\frac{\theta_-+\vartheta_-}{\sigma}\right] \\ &+\left(\frac{B}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}\mathcal{N}(\eta e_-)\left[r_d(1-\omega)+(r_d-r_f)\frac{\theta_--\vartheta_-}{\sigma}\right] \\ &+\eta\left(\frac{B}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}n(e_+)\left[-\frac{e_-}{\tau}+\frac{r_d-r_f}{\sigma\sqrt{\tau}}\right] \\ &+\eta\left(\frac{B}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}n(e_-)\left[\frac{e_+}{\tau}+\frac{r_d-r_f}{\sigma\sqrt{\tau}}\right]\Bigg\}. \end{aligned} \tag{45}$$

Vega To compute vega we use the identities

$$\frac{\partial\theta_-}{\partial\sigma} = -\frac{\theta_+}{\sigma}, \tag{46}$$

$$\frac{\partial\vartheta_-}{\partial\sigma} = -\frac{\theta_-\theta_+}{\sigma\vartheta_-}, \tag{47}$$

$$\frac{\partial e_\pm}{\partial\sigma} = \pm\frac{\log\frac{B}{x}}{\sigma^2\sqrt{\tau}}+\frac{\theta_-\theta_+}{\sigma\vartheta_-}\sqrt{\tau}, \tag{48}$$

$$A_\pm \triangleq \frac{\partial}{\partial\sigma}\frac{\theta_-\pm\vartheta_-}{\sigma} = -\frac{1}{\sigma^2}\left[\theta_++\theta_-\pm\left(\frac{\theta_-\theta_+}{\vartheta_-}+\vartheta_-\right)\right], \tag{49}$$

and obtain

$$\begin{aligned} v_\sigma(t,x) &= Re^{-\omega r_d \tau} \cdot \\ &\quad \left\{ \left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} \left[\mathcal{N}(-\eta e_+) A_+ \log\left(\frac{B}{x}\right) - \eta n(e_+) \frac{\partial e_+}{\partial \sigma}\right] \right. \\ &\quad \left. + \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} \left[\mathcal{N}(\eta e_-) A_- \log\left(\frac{B}{x}\right) + \eta n(e_-) \frac{\partial e_-}{\partial \sigma}\right] \right\}. \end{aligned} \tag{50}$$

1.2.7 Knock-out probability

The risk-neutral probability of knocking out is given by

$$\mathbb{P}[\tau_B \leq T] = \mathbb{E}\left[I_{\{\tau_B \leq T\}}\right] = \frac{1}{R} e^{r_d T} v(0, S_0). \tag{51}$$

1.2.8 Properties of the first hitting time τ_B

As derived, e.g., in [26], the first hitting time

$$\tilde{\tau} \triangleq \inf\{t \geq 0 : \theta t + W(t) = x\} \tag{52}$$

of a Brownian motion with drift θ and hit level $x > 0$ has the density

$$\mathbb{P}[\tilde{\tau} \in dt] = \frac{x}{t\sqrt{2\pi t}} \exp\left\{-\frac{(x - \theta t)^2}{2t}\right\} dt, \quad t > 0, \tag{53}$$

the cumulative distribution function

$$\mathbb{P}[\tilde{\tau} \leq t] = \mathcal{N}\left(\frac{\theta t - x}{\sqrt{t}}\right) + e^{2\theta x} \mathcal{N}\left(\frac{-\theta t - x}{\sqrt{t}}\right), \quad t > 0, \tag{54}$$

the Laplace-transform

$$\mathbb{E} e^{-\alpha \tilde{\tau}} = \exp\left\{x\theta - x\sqrt{2\alpha + \theta^2}\right\}, \quad \alpha > 0,\ x > 0, \tag{55}$$

and the property

$$\mathbb{P}[\tilde{\tau} < \infty] = \left\{ \begin{array}{ll} 1 & \text{if } \theta \geq 0 \\ e^{2\theta x} & \text{if } \theta < 0 \end{array} \right. . \tag{56}$$

For upper barriers $B > S_0$ we can now rewrite the first passage time τ_B as

$$\begin{aligned} \tau_B &= \inf\{t \geq 0 : S_t = B\} \\ &= \inf\left\{t \geq 0 : W_t + \theta_- t = \frac{1}{\sigma} \log\left(\frac{B}{S_0}\right)\right\}. \end{aligned} \tag{57}$$

The density of τ_B is hence

$$\mathbb{P}[\tilde{\tau_B} \in dt] = \frac{\frac{1}{\sigma} \log\left(\frac{B}{S_0}\right)}{t\sqrt{2\pi t}} \exp\left\{-\frac{\left(\frac{1}{\sigma} \log\left(\frac{B}{S_0}\right) - \theta_- t\right)^2}{2t}\right\}, \quad t > 0. \tag{58}$$

1.2.9 Derivation of the value function

Using the density (58) the value of the paid-at-end ($\omega = 1$) upper rebate ($\eta = -1$) option can be written as

$$\begin{aligned} v(T, S_0) &= Re^{-r_d T}\mathbb{E}\left[I_{\{\tau_B \le T\}}\right] && (59) \\ &= Re^{-r_d T}\int_0^T \frac{\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)}{t\sqrt{2\pi t}}\exp\left\{-\frac{(\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right) - \theta_- t)^2}{2t}\right\} dt. \end{aligned}$$

To evaluate this integral, we introduce the notation

$$e_\pm(t) \triangleq \frac{\pm\log\frac{S_0}{B} - \sigma\theta_- t}{\sigma\sqrt{t}} \tag{60}$$

and list the properties

$$\begin{aligned} e_-(t) - e_+(t) &= \frac{2}{\sqrt{t}}\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right), && (61) \\ n(e_+(t)) &= \left(\frac{B}{S_0}\right)^{-\frac{2\theta_-}{\sigma}} n(e_-(t)), && (62) \\ \frac{\partial e_\pm(t)}{\partial t} &= \frac{e_\mp(t)}{2t}. && (63) \end{aligned}$$

We evaluate the integral in (59) by rewriting the integrand in such a way that the coefficients of the exponentials are the inner derivatives of the exponentials using properties (61),(62) and (63),

$$\begin{aligned} &\int_0^T \frac{\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)}{t\sqrt{2\pi t}}\exp\left\{-\frac{(\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right) - \theta_- t)^2}{2t}\right\} dt \\ = &\ \frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)\int_0^T \frac{1}{t^{(3/2)}} n(e_-(t))\,dt \\ = &\int_0^T \frac{1}{2t} n(e_-(t))[e_-(t) - e_+(t)]\,dt \\ = &\ -\int_0^T n(e_-(t))\frac{e_+(t)}{2t} + \left(\frac{B}{S_0}\right)^{\frac{2\theta_-}{\sigma}} n(e_+(t))\frac{e_-(t)}{2t}\,dt \\ = &\left(\frac{B}{S_0}\right)^{\frac{2\theta_-}{\sigma}}\mathcal{N}(e_+(T)) + \mathcal{N}(-e_-(T)). && (64) \end{aligned}$$

The computation for lower barriers ($\eta = 1$) is similar.

1.2.10 deviation of Market prices from the TV

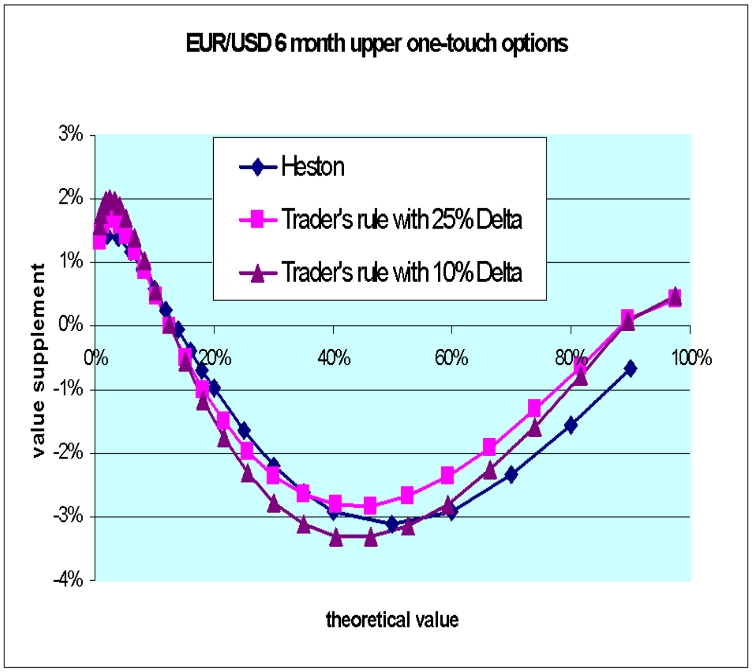


Figure 1: Comparison of value adjustments of one-touch options in Heston's model

1.3 Double no-touch options

1.3.1 payoff

$$I_{\{L \le \min_{[0,T_e]} S_t < \max_{[0,T_e]} S_t \le H\}} \tag{65}$$

1.3.2 incentive

Forward Plus contracts Expected low volatility will lead to profit for the client

1.3.3 Pricing

On $[t, \tau]$, the price of the option is

$$v(t) = e^{r_t^{T_p} T_p} \mathbb{E}^t \left[e^{-r_t^{T_d}(T_d - t)} I_{\{L \le \min_{[0,T_e]} S_t < \max_{[0,T_e]} S_t \le H\}} \right], \tag{66}$$

on $[\tau, T_d]$,

$$v(t) = e^{r_t^{T_p} T_p - r_d(T_d - t)} I_{\{L \le \min_{[0,T_e]} S_t < \max_{[0,T_e]} S_t \le H\}}. \tag{67}$$

To compute the expectation, let us introduce the stopping time

$$\tau \triangleq \min\left\{\inf\left\{t \in [0, T_e] | S_t = L \text{ or } S_t = H\right\}, T_e\right\} \tag{68}$$

and the notation

$$\begin{aligned}
\tilde{\theta} &\triangleq \frac{r_d - r_f - \frac{1}{2}\sigma^2}{\sigma} && (69)\\
\tilde{h} &\triangleq \frac{1}{\sigma}\ln\frac{H}{S_t} && (70)\\
\tilde{l} &\triangleq \frac{1}{\sigma}\ln\frac{L}{S_t} && (71)\\
\theta &\triangleq \tilde{\theta}\sqrt{T_e - t} && (72)\\
h &\triangleq \tilde{h}/\sqrt{T_e - t} && (73)\\
l &\triangleq \tilde{l}/\sqrt{T_e - t} && (74)\\
y_n &\triangleq 2n(h-l) - \theta && (75)\\
n_T(x) &\triangleq \frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{x^2}{2T}\right). && (76)
\end{aligned}$$

The joint distribution of the maximum and the minimum of a Brownian motion can be taken from [23] and is given by

$$\mathbb{P}\left[\tilde{l} \le \min_{[0,T]} W_t < \max_{[0,T]} W_t \le \tilde{h}\right] = \int_{\tilde{l}}^{\tilde{h}} k_T(x)\,dx \tag{77}$$

with

$$k_T(x) = \sum_{n=-\infty}^{\infty}\left[n_T(x + 2n(\tilde{h} - \tilde{l})) - n_T(x - 2\tilde{h} + 2n(\tilde{h} - \tilde{l}))\right]. \tag{78}$$

Hence the joint density of the maximum and the minimum of a Brownian motion with drift $\tilde{\theta}$, $W_t^{\tilde{\theta}} \triangleq W_t + \tilde{\theta}t$, is given by

$$k_T^{\tilde{\theta}}(x) = k_T(x)\exp\left\{\tilde{\theta}x - \frac{1}{2}\tilde{\theta}^2 T\right\}. \tag{79}$$

We obtain for the price of the option on $[t,\tau]$

$$\begin{aligned}
v(t) &= e^{r_t^{T_p}T_p - r_d(T_d-t)} I_{\{L \le \min_{[0,T_e]} S_t < \max_{[0,T_e]} S_t \le H\}}\\
&= e^{r_t^{T_p}T_p - r_d(T_d-t)} I_{\{\tilde{l} \le \min_{[0,T_e]} W_t^{\tilde{\theta}} < \max_{[0,T_e]} W_t^{\tilde{\theta}} \le \tilde{h}\}}\\
&= e^{r_t^{T_p}T_p - r_d(T_d-t)} \int_{\tilde{l}}^{\tilde{h}} k_{(T_e-t)}^{\tilde{\theta}}(x)dx \qquad (80)\\
&= e^{r_t^{T_p}T_p - r_t^{T_d}(T_d-t)}\\
&\quad \cdot \sum_{n=-\infty}^{\infty}\Big[e^{-2n\theta(h-l)}\left\{\mathcal{N}(h+y_n) - \mathcal{N}(l+y_n)\right\}\\
&\quad - e^{-2n\theta(h-l)+2\theta h}\left\{\mathcal{N}(h-2h+y_n) - \mathcal{N}(l-2h+y_n)\right\}\Big]
\end{aligned}$$

and on $[\tau, T_d]$

$$v(t) = e^{r_t^{T_p}T_p - r_d(T_d-t)} I_{\{L \le \min_{[0,T_e]} S_t < \max_{[0,T_e]} S_t \le H\}}. \tag{81}$$

1.4 Single Barrier Options

model

$$dS(t) = (r_d - r_f)S(t)\,dt + \sigma S(t)\,dW(t), \quad S(0) > 0, \tag{82}$$

- The *domestic interest rate* $r_d \in \mathbb{R}$,
- the *foreign interest rate* or *continuous dividend rate* $r_f \in \mathbb{R}$,
- the *volatility* $\sigma > 0$
- and the planning horizon $T > 0$

are assumed to be constant. The process $(W(t); 0 \le t \le T)$ is a Brownian motion under a probability measure $\mathbb{P}$ which is *risk-neutral,* i.e., is chosen so that the stock has mean rate of return $r_d - r_f$.

payoff For knock-out options we consider the payoff

$$[\phi(S_T - K)]^+ I_{\{\eta S_t > \eta B, 0 \le t \le T\}}, \tag{83}$$

- ϕ takes the values $+1$ for a call and -1 for a put,
- η takes the values $+1$ if the barrier B is approached from above (down-and-out) and -1 if the barrier is approached from below (up-and-out).
- The strike is denoted by K.

To price kick-in options paying

$$[\phi(S_T - K)]^+ I_{\{\min[\eta S_t] < \eta B\}} \tag{84}$$

we use the fact that

$$\text{kick-in} + \text{knock-out} = \text{vanilla}. \tag{85}$$

abbreviations

- t: running time (in years)
- $\tau \triangleq T - t$: time to expiration (in years)
- $\theta_\pm \triangleq \frac{r_d - r_f}{\sigma} \pm \frac{\sigma}{2}$
- $\mu \triangleq \sigma\theta_-$
- $n(t) \triangleq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$
- $\mathcal{N}(x) \triangleq \int_{-\infty}^{x} n(t)\, dt$
- $\lambda = 1 + \frac{\mu}{\sigma^2}$
- $a = \frac{\mu}{\sigma^2}$
- $p = (\mu + \sigma^2)\tau$
- $X = \frac{\log(\frac{x}{K}) + p}{\sigma\sqrt{\tau}}$
- $x_1 = \frac{\log(\frac{x}{B}) + p}{\sigma\sqrt{\tau}}$
- $y = \frac{\log(\frac{B^2}{xK}) + p}{\sigma\sqrt{\tau}}$
- $y_1 = \frac{\log(\frac{B}{x}) + p}{\sigma\sqrt{\tau}}$

- $w = \frac{\log(\frac{B}{x})+m\sigma^2\tau}{\sigma\sqrt{\tau}}$
- $z = 1 - \frac{K}{B}$
- $d = e^{-r_d\tau}$
- $f = e^{-r_f\tau}$

description via PDE We can describe a barrier option's value function as a solution to a partial differential equation setup. Let $v(t,x)$ denote the value of the option at time t when the underlying is at x. Then $v(t,x)$ is the solution of

$$\begin{aligned} v_t + (r_d - r_f)xv_x + \frac{1}{2}\sigma^2x^2v_{xx} - r_dv &= 0, \\ t \in [0,T],\ \eta x &\geq \eta B, \quad (86) \\ v(T,x) &= [\phi(x-K)]^+, \\ \eta x &\geq \eta B, \quad (87) \\ v(t,B) &= 0,\ t \in [0,T]. \quad (88) \end{aligned}$$

1.4.1 value

Evaluating the integral produces four terms. We first provide the four terms and then explain how they are used to find the value function. Note that always the last of the A_i is valid, others are only for temporary use.

$$A_1 = \phi x f \mathcal{N}(\phi X) - \phi K d \mathcal{N}(\phi(X - \sigma\sqrt{\tau})) \quad (89)$$

$$A_2 = \phi x f \mathcal{N}(\phi x_1) - \phi K d \mathcal{N}(\phi(x_1 - \sigma\sqrt{\tau})) \quad (90)$$

$$A_3 = \phi \left(\frac{B}{x}\right)^{2\lambda-2}\left[xf\left(\frac{B}{x}\right)^2\mathcal{N}(\eta y) - Kd\mathcal{N}(\eta(y-\sigma\sqrt{\tau}))\right] \quad (91)$$

$$\begin{aligned} A_4 &= \phi\frac{-2\mu}{\sigma^2 x}\left(\frac{B}{x}\right)^{2\lambda-2}\left[xf\left(\frac{B}{x}\right)^2\mathcal{N}(\eta y_1) - Kd\mathcal{N}(\eta(y_1-\sigma\sqrt{\tau}))\right] \\ &\quad -\phi\left(\frac{B}{x}\right)^{2\lambda} f\mathcal{N}(\eta y_1) - \phi\eta f\left(\frac{B}{x}\right)^{2\lambda} n(y_1)z/\sigma\sqrt{\tau} \quad (92) \end{aligned}$$

option type	ϕ	η	in/out	reverse	combination
standard up-and-in call	$+1$	-1	-1	$K > B$	A_1
reverse up-and-in call	$+1$	-1	-1	$K \leq B$	$A_2 - A_3 + A_4$
reverse up-and-in put	-1	-1	-1	$K > B$	$A_1 - A_2 + A_4$
standard up-and-in put	-1	-1	-1	$K \leq B$	A_3
standard down-and-in call	$+1$	$+1$	-1	$K > B$	A_3
reverse down-and-in call	$+1$	$+1$	-1	$K \leq B$	$A_1 - A_2 + A_4$
reverse down-and-in put	-1	$+1$	-1	$K > B$	$A_2 - A_3 + A_4$
standard down-and-in put	-1	$+1$	-1	$K \leq B$	A_1
standard up-and-out call	$+1$	-1	$+1$	$K > B$	0
reverse up-and-out call	$+1$	-1	$+1$	$K \leq B$	$A_1 - A_2 + A_3 - A_4$
reverse up-and-out put	-1	-1	$+1$	$K > B$	$A_2 - A_4$
standard up-and-out put	-1	-1	$+1$	$K \leq B$	$A_1 - A_3$
standard down-and-out call	$+1$	$+1$	$+1$	$K > B$	$A_1 - A_3$
reverse down-and-out call	$+1$	$+1$	$+1$	$K \leq B$	$A_2 - A_4$
reverse down-and-out put	-1	$+1$	$+1$	$K > B$	$A_1 - A_2 + A_3 - A_4$
standard down-and-out put	-1	$+1$	$+1$	$K \leq B$	0

1.4.2 Greeks

delta

$$\begin{aligned}
A_1 &= \phi f \mathcal{N}(\phi X) \qquad (93)\\
A_2 &= \phi f \mathcal{N}(\phi x_1) + f n(x_1) z / \sigma\sqrt{\tau}\\
A_3 &= \phi \frac{-2\mu}{\sigma^2 x}\left(\frac{B}{x}\right)^{2\lambda-2}\left[x f\left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y) - K d \mathcal{N}(\eta(y-\sigma\sqrt{\tau}))\right]\\
&\quad -\phi\left(\frac{B}{x}\right)^{2\lambda} f \mathcal{N}(\eta y)\\
A_4 &= \phi\left(\frac{B}{x}\right)^{2\lambda-2}\left[x f\left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y_1) - K d \mathcal{N}(\eta(y_1-\sigma\sqrt{\tau}))\right]
\end{aligned}$$

gamma

$$
\begin{aligned}
A_1 &= fn(X)/(x\sigma\sqrt{\tau}) \\
A_2 &= fn(x_1)/(x\sigma\sqrt{\tau})(1 - zx_1/\sigma\sqrt{\tau}) \\
C_3 &= \phi\left(\frac{B}{x}\right)^{2\lambda-2}\left[xf\left(\frac{B}{x}\right)^2\mathcal{N}(\eta y) - Kd\mathcal{N}(\eta(y-\sigma\sqrt{\tau}))\right] \\
B_3 &= \frac{-2\mu}{\sigma^2 x}C_3 - \phi\left(\frac{B}{x}\right)^{2\lambda}f\mathcal{N}(\eta y) \\
A_3 &= \frac{2\mu}{\sigma^2 x}(C_3/x - B_3) + \phi f B^{2\lambda}/x^{2\lambda+1}\left[2\lambda\mathcal{N}(\eta y) + \eta n(y)/\sigma\sqrt{\tau}\right] \\
C_4 &= \phi\left(\frac{B}{x}\right)^{2\lambda-2}\left[xf\left(\frac{B}{x}\right)^2\mathcal{N}(\eta y_1) - Kd\mathcal{N}(\eta(y_1-\sigma\sqrt{\tau}))\right] \\
B_4 &= \frac{-2\mu}{\sigma^2 x}C_4 - \phi\left(\frac{B}{x}\right)^{2\lambda}f\mathcal{N}(\eta y_1) - \phi\eta f\left(\frac{B}{x}\right)^{2\lambda}n(y_1)z/\sigma\sqrt{\tau} \\
A_4 &= \frac{2\mu}{\sigma^2 x}(C_4/x - B_4) + \phi f B^{2\lambda}/x^{2\lambda+1}\left[2\lambda\mathcal{N}(\eta y_1) + \eta n(y_1)/\sigma\sqrt{\tau}\right] \\
&\quad +\phi\eta f z n(y_1)\left(\frac{B}{x}\right)^{2\lambda}/(x\sigma\sqrt{\tau})(2\lambda - y_1/\sigma\sqrt{\tau})
\end{aligned}
\tag{94}
$$

theta

$$
\begin{aligned}
A_1 &= -\frac{1}{2}\sigma x f n(X)/\sqrt{\tau} + \phi x f\mathcal{N}(\phi X)r_f - \phi Kd\mathcal{N}(\phi(X-\sigma\sqrt{\tau}))r_d \\
A_2 &= -\frac{1}{2}\sigma x f n(x_1)K/(B\sqrt{\tau}) + \phi x f\mathcal{N}(\phi x_1)r_f - \phi Kd\mathcal{N}(\phi(x_1-\sigma\sqrt{\tau}))r_d \\
&\quad -xfn(x_1)zy_1/(2\tau) \\
A_3 &= -\phi\left(\frac{B}{x}\right)^{2\lambda}xf\eta n(y)\frac{1}{2}\sigma/\sqrt{\tau} \\
&\quad +\phi\left(\frac{B}{x}\right)^{2\lambda-2}\left[r_f xf\left(\frac{B}{x}\right)^2\mathcal{N}(\eta y) - r_d Kd\mathcal{N}(\eta(y-\sigma\sqrt{\tau}))\right] \\
A_4 &= -\phi\left(\frac{B}{x}\right)^{2\lambda}xf\eta n(y_1)\left[x_1/(2\tau)z + \frac{1}{2}\sigma K/(\sqrt{\tau}B)\right] \\
&\quad +\phi\left(\frac{B}{x}\right)^{2\lambda-2}\left[r_f xf\left(\frac{B}{x}\right)^2\mathcal{N}(\eta y_1) - r_d Kd\mathcal{N}(\eta(y_1-\sigma\sqrt{\tau}))\right]
\end{aligned}
\tag{95}
$$

vega

$$
\begin{aligned}
A_1 &= xfn(X)\sqrt{\tau} \qquad (96)\\
A_2 &= xfn(x_1)(\sqrt{\tau} - x_1 z/\sigma)\\
B_3 &= \phi\left(\frac{B}{x}\right)^{2\lambda-2}\left[xf\left(\frac{B}{x}\right)^{2}\mathcal{N}(\eta y) - Kd\mathcal{N}(\eta(y-\sigma\sqrt{\tau}))\right]\\
A_3 &= \frac{-4}{\sigma^3}\log\left(\frac{B}{x}\right)(r_d - r_f)B_3 + \phi\left(\frac{B}{x}\right)^{2\lambda} xf\eta n(y)\sqrt{\tau}\\
B_4 &= \phi\left(\frac{B}{x}\right)^{2\lambda-2}\left[xf\left(\frac{B}{x}\right)^{2}\mathcal{N}(\eta y_1) - Kd\mathcal{N}(\eta(y_1-\sigma\sqrt{\tau}))\right] \qquad (97)\\
A_4 &= \frac{-4}{\sigma^3}\log\left(\frac{B}{x}\right)(r_d - r_f)B_4\\
&\quad +\phi\left(\frac{B}{x}\right)^{2\lambda} xf\eta n(y_1)\left[(\sqrt{\tau} - y_1/\sigma)z + \frac{K}{B}\sqrt{\tau}\right]
\end{aligned}
$$

joint density for final time value and running extremum As derived, e.g., in [26], joint density $f(x,y)$ for a Brownian motion with drift and its running extremum ($\eta = +1$ for a maximum and $\eta = -1$ for a minimum)

$$
\left(W(T) + \theta_{-}T, \eta \min_{0\le t\le T}[\eta(W(t) + \theta_{-}t)]\right) \qquad (98)
$$

$$
f(x,y) = -\eta e^{\theta_{-}x - \frac{1}{2}\theta_{-}^{2}T}\frac{2(2y-x)}{T\sqrt{2\pi T}}\exp\left\{-\frac{(2y-x)^2}{2T}\right\}, \qquad (99)
$$
$$
\eta y \le \min(0, \eta x).
$$

derivation of the value function Using the density (99) the value of a barrier option can be written as the following integral

$$
\begin{aligned}
&\text{barrier}(S_0, \sigma, r_d, r_f, K, B, T) \qquad (100)\\
= \;& e^{-r_d T}\mathbb{E}\left[[\phi(S_T - K)]^{+} I_{\{\eta S_t > \eta B, 0\le t\le T\}}\right]\\
= \;& e^{-r_d T}\int_{x=-\infty}^{x=-\infty}\int_{\eta y\le\min(0,\eta x)}[\phi(S_0 e^{\sigma x} - K)]^{+} I_{\{\eta y > \eta\frac{1}{\sigma}\log\frac{B}{S_0}\}} f(x,y)\,dy\,dx.
\end{aligned}
$$

Further details on how to evaluate this integral can be found in [26].

1.5 Double barrier options

payoff

$$
(\phi(S_{T_e} - K))^{+} I_{\{L < \min_{[0,T_e]} S_t \le \max_{[0,T_e]} S_t < H\}}, \qquad (101)
$$

1.5.1 Pricing

The distribution of S_{T_e} conditioned on not having reached the upper barrier H and the lower barrier L is

$$\begin{aligned}
& e^{-\frac{1}{2}\lambda^2(T_e-t)+\frac{\lambda}{\sigma}\ln\frac{S_{T_e}}{S_t}} \times \\
& \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \left[\exp\left(-\frac{1}{2\sigma^2(T_e-t)} \left(\ln\frac{S_{T_e}}{S_t} + 2n\ln\frac{H}{L} \right)^2 \right) \right. \\
& \left. - \exp\left(-\frac{1}{2\sigma^2(T_e-t)} \left(\ln\frac{H^2}{S_{T_e}S_t} + 2n\ln\frac{H}{L} \right)^2 \right) \right] I_{\{L<S_{T_e}<H\}}
\end{aligned} \tag{102}$$

with

$$\lambda \triangleq \frac{\mu}{\sigma} - \frac{\sigma}{2}. \tag{103}$$

To price the option, let us introduce the stopping time

$$\tau \triangleq \min\{\inf\{t \in [0,T_e] | S_t = L \text{ or } S_t = H\}, T_e\} \tag{104}$$

The price of the option on $[t,\tau]$ is

$$v(t) = e^{-r_{T_e}(T_e-t)}\mathbb{E}^t\left[\left(\phi\left(S_T - K\right)\right)^+ I_{\{L<\min_{[t,T_e]} S_s \le \max_{[t,T_e]} S_s < H\}} \right] \tag{105}$$

and on $[\tau, T_d]$

$$v(t) = 0. \tag{106}$$

1.5.2 Difference to two barrier options

Two single barrier options with barriers L and H are not the same as one double barrier option with the same barriers. In fact double barriers are cheaper.

2 Difficulties created by the characteristics of barrier options

Example: *knock-out call*

$$(S(T)-K)^+ I_{\{\max_{0\le t\le T} S(t)<B\}} \tag{107}$$

$$0 < K < B \tag{108}$$

This call "knocks out" in the money, which makes the implementation of the Black-Scholes hedging strategy difficult because it has large delta and gamma values near the barrier near expiration. A trader who is delta-hedging a short position in this option would take large short positions in the underlying asset and make large adjustments to this position.

Related work

Cvitanić & Karatzas and El Karoui & Quenez.

- Characterization of the *upper hedging price*, as a minimization problem: Minimal initial wealth to super-replicate the payoff.
- Dual problem, which is one of maximization over changes of measure, and the equivalence of the two problems was shown in [8] and [12].

Broadie, Cvitanić & Soner [5] showed that for a contingent claim whose payoff at expiration is a function of the final value of a single, geometric Brownian motion, the dual problem can be solved in two steps.

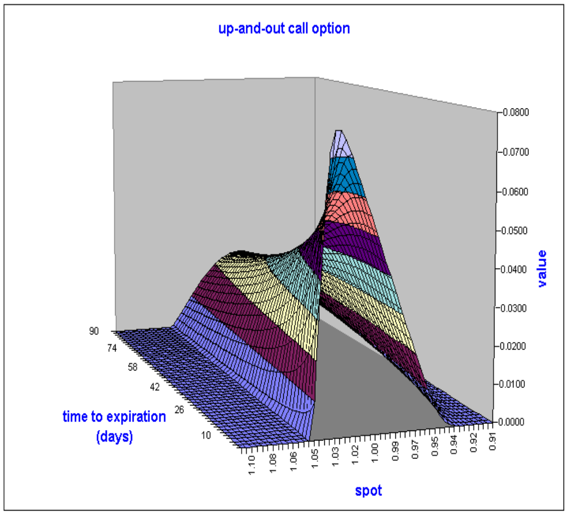


Figure 2: value of an up-and-out call option $v(t, x)$ given by (109) with strike $K = 0.95$, knock-out barrier $B = 1.05$ and maturity $T = 90/365$. We used the interest rates $r_d = 5\%$, $r_f = 0\%$ and volatility $\sigma = 10\%$

1. computes the *face-lift,* of the payoff function (see (115) below).
2. One next prices the contingent claim whose payoff at the final time is the face-lifted version of the original payoff. One does this using the usual risk-neutral pricing formula, i.e., without regard to the portfolio constraint.

Schmock, Shreve and Wystup [27] extended this idea to the case of path-dependent options with a lower bound on the hedging portfolio. They provide a reformulation of the dual problem of [8] and [12] so that the solution can often be obtained by inspection.

Cvitanic, Pham, Touzi and Bouchard The role of upper hedging prices in the presence of stochastic volatility and/or transaction costs in [3], [9], [10], [29].

Soner and Touzi Gamma constraints in [28].

Karatzas and Kou Lower hedging prices in [16]

Karatzas and Kou [17] perpetual American options using similar methodology

Föllmer and Leukert Quantile Hedging: Since exact super-replication is in general too expensive, many authors including [13] examine hedging strategies which succeed with high probability.

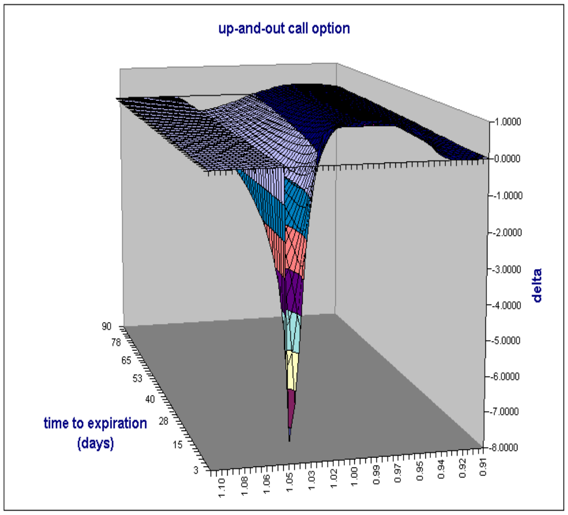


Figure 3: delta of an up-and-out call option $v_x(t,x)$ given by (109) with strike $K = 0.95$, knock-out barrier $B = 1.05$ and maturity $T = 90/365$. We used the interest rates $r_d = 5\%$, $r_f = 0\%$ and volatility $\sigma = 10\%$

up-and-out call formula Let $\mathcal{N}$ denote the standard normal distribution function. Using formula (99), the value $v(t,x)$ of an up-and-out call option is for $t \in [0,T)$ and $x \in (0,B]$

$$\begin{aligned} v(t,x) = {} & xe^{-r_f\tau}\big[\mathcal{N}(b-\theta_+) - \mathcal{N}(k-\theta_+)\big] \\ & + xe^{-r_f\tau+2b\theta_+}\big[\mathcal{N}(b+\theta_+) - \mathcal{N}(2b-k+\theta_+)\big] \\ & - Ke^{-r_d\tau}\big[\mathcal{N}(b-\theta_-) - \mathcal{N}(k-\theta_-)\big] \\ & - Ke^{-r_d\tau+2b\theta_-}\big[\mathcal{N}(b+\theta_-) - \mathcal{N}(2b-k+\theta_-)\big], \end{aligned} \tag{109}$$

where $b \triangleq \frac{1}{\sigma\sqrt{\tau}}\log\frac{B}{x}$, $k \triangleq \frac{1}{\sigma\sqrt{\tau}}\log\frac{K}{x}$, $r \triangleq r_d - r_f$ and $\theta_\pm = (\frac{r}{\sigma} \pm \frac{\sigma}{2})\sqrt{\tau}$.

2.1 Infinite deltas and gammas

We have

- $v(t,B) = 0$ for $0 \le t \le T$.
- For $0 < x \le B$, as $t \uparrow T$, we obtain from (109) that $v(t,x)$ approaches the discontinuous limit $v(T,x) = (x-K)^+ I_{\{x<B\}}$.

Consequently, for x near B and t near T, the "delta" $v_x(t,x)$ and "gamma" $v_{xx}(t,x)$ of this option become large in absolute value as illustrated in Figures 3 and 4.

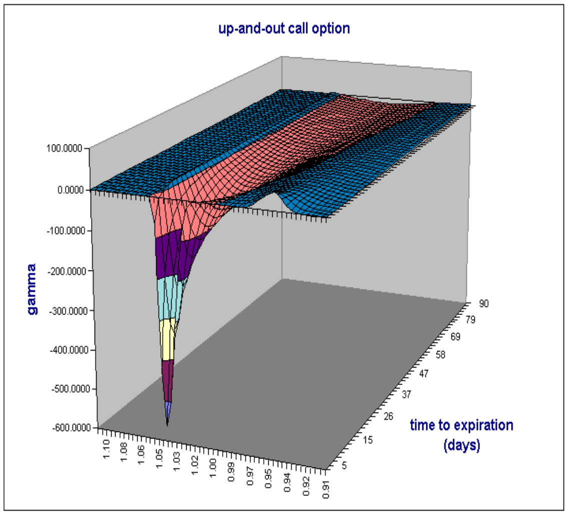


Figure 4: gamma of an up-and-out call option $v_{xx}(t,x)$ given by (109) with strike $K = 0.95$, knock-out barrier $B = 1.05$ and maturity $T = 90/365$. We used the interest rates $r_d = 5\%$, $r_f = 0\%$ and volatility $\sigma = 10\%$

3 Exotic barrier options to minimise such difficulties

The risk of loss and the hedging problem of barrier options have been recognized by trading practitioners as well as academics. There are various ways to limit this risk and this hedging difficulty as for example

Include rebates , see Section 3.1.

Step Options . Modify the knock-out regulation as follows. The final payoff looses its value at a rate proportional to the total time the stock spends above the barrier. Such barrier options are also called *soft barrier options* and are discussed, e.g., in [15] or [20].

Parasian/Parisian Barriers . Modify the knock-out regulation as follows. The final payoff looses its value only if the stock spends a pre-specified time interval above the barrier in total / in sequence. Such options are discussed in [6] and [7].

3.1 Rebates

kick-in option . a sum R is paid at expiration by the seller of the option to the holder of the option if the option failed to kick in during its lifetime.

knock-out option . a sum R is paid by the seller of the option to the holder of the option, if the option knocks out. There are two kinds of agreements in the knock-out case:

- **paid at expiration** T . the boundary condition of the Black-Scholes differential equation is $v(t,B) = Re^{-r_d(T-t)}$,
- **paid at first hitting time** . the corresponding boundary condition becomes $v(t,B) = R$.

Including such rebate features makes hedging easier, which could be one of the reasons they were invented.

3.2 Step options

For details here see the work on *occupation time derivatives* by Vadim Linetzky [20] and Andreas Pechtl [21].

3.3 Parisian options

The development of variants of the classic concept includes the introduction of barriers which are not monitored every day 24 hours/day of the lifetime of the option but rather at some singular points in time like the work-daily OPTREF-fixing at 13:00.

The impact on the price can be rather dramatic. Consider the product:

- put EUR / call USD 1y
- strike 0.90
- barrier down&out 0.80
- barrier up&out 1.00

with Spot 0.8450, USD rate 6.0%, EUR rate 4.80% and volatility of 14%. We vary the number of fixings from daily to monthly. Even for a daily fixing for a period of one year the price is 33% higher.

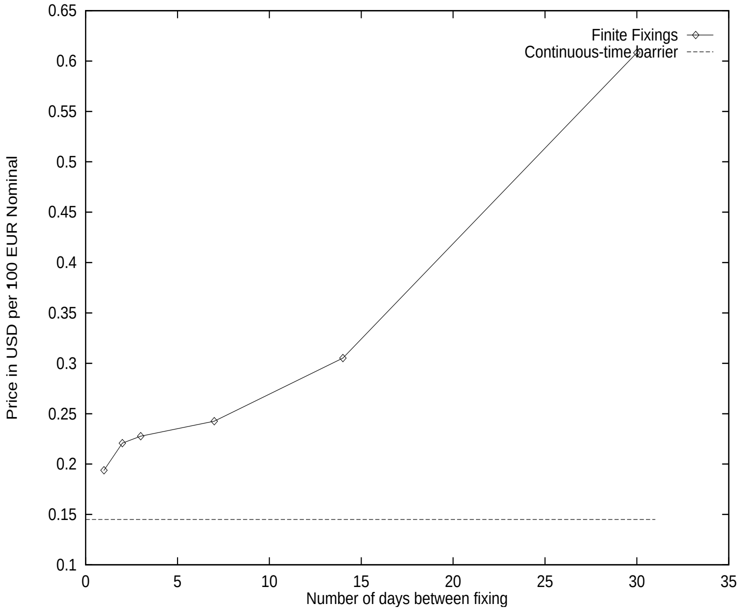


Figure 5: Comparison between a continuously monitored barrier and finitely many fixings

Parasian and Parisian

definition To knock-out the option it is necessary to be knocked out for several fixings, either *in total* (parasian) or *in sequence* (parisian).

special case: one knock-out day The Parisian barrier option with one knock-out day is the same product as the barrier option with one fixing per day.

behavior We show the behavior for a varying number of Parisian and Parasian knock-out days. The behaviour for a few knock-out days is very different, where the Parisian barrier is always more expensive than the Parasian. For many knock-out days the prices of both products are rather similar.

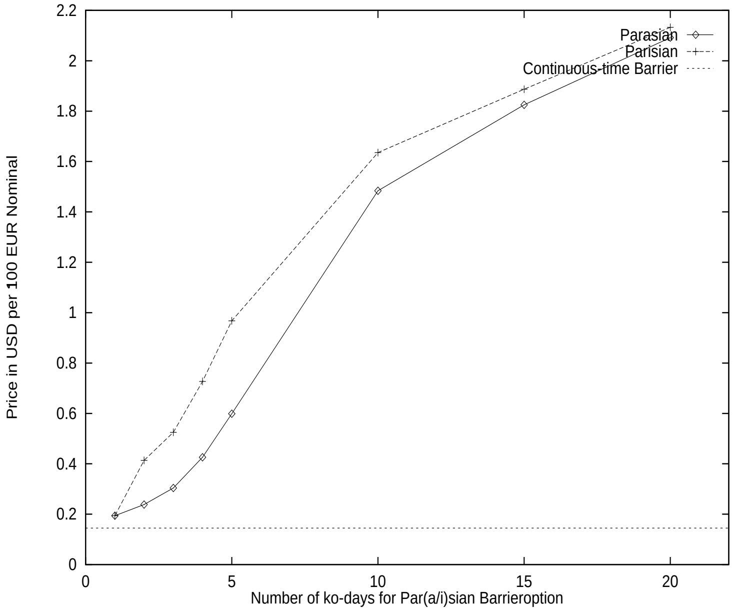


Figure 6: Comparison of option price for a continuously monitored barrier, a Parasian barrier and a Parisian barrier

common goal:
- Lift the option's value at the barrier
- The holder does not face sudden loss of the entire option contract.
- The seller then has a smaller delta and gamma for the hedge.

our approach:
- Don't change the contract
- constrain the size of the short position allowed for hedging a short option position
- incorporate the cost of this constraint into the price of the option.

4 Applying static hedging to barrier options

Several authors, including [1], [4], and [11], have suggested static hedges for dangerous exotic options. This means to compose a portfolio of vanilla options with the same payoff as the barrier option. The following problems occur.

large nominals

liquidity of vanilla options is not guaranteed when closing the position

hard to combine with stochastic volatility

4.1 Coping with the effects of liquidity

Look at the example of Derman, Ergener and Kani [11].

- 3 month up-and-out call
- spot = 1.78, strike = 1.70, barrier = 1.85
- domestic rate = 3.29%, foreign rate = 5.72%.
- the analytic fair value is 0.0196 (vol = 10.9%).
- Figure 7 shows the term structure of volatility.
- Figure 8 shows the portfolio of call options for a static hedge. The fair value of all these vanillas is 0.0190, but taking the term structure of volatility into account the value of the hedge is 0.0364, which is substantially higher and not tradable.
- Figure 9 shows the value of the portfolio of call options at the barrier. Ideally we would have a straight line at zero.

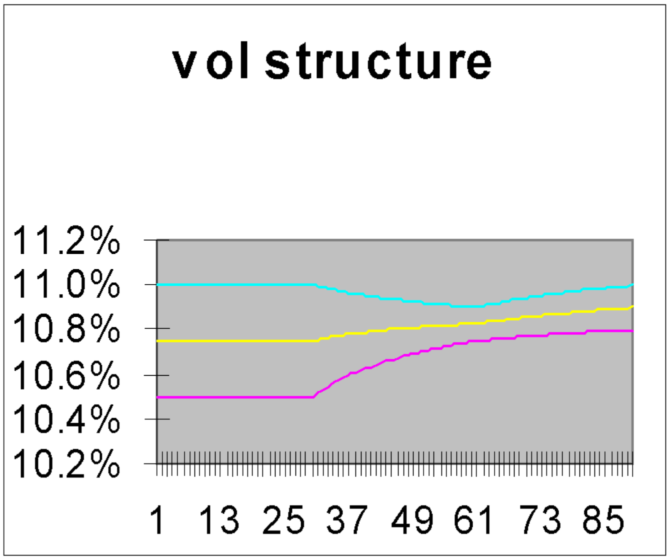
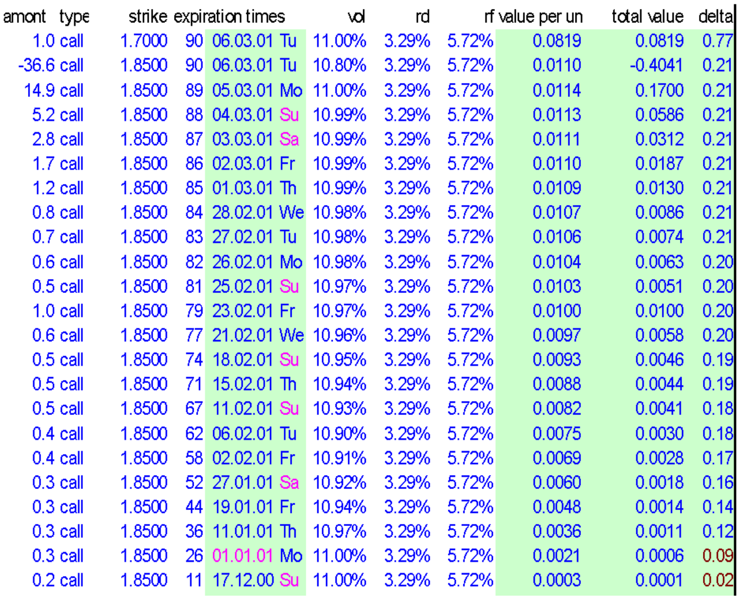


Figure 7: Term structure of volatility, bid and aks

Problems:

- portfolio is too large
- nominals of calls are large
- most of the call options are low deltas, which can cause a liquidity problem or make the hedge very expensive
- the replication still involves risk of losses at the time of closing the hedging position
- biggest drawback: the static hedge changes with changing volatility



amont	type	strike	expiration times			vol	rd	rf	value per un	total value	delta
1.0	call	1.7000	90	06.03.01	Tu	11.00%	3.29%	5.72%	0.0819	0.0819	0.77
-36.6	call	1.8500	90	06.03.01	Tu	10.80%	3.29%	5.72%	0.0110	-0.4041	0.21
14.9	call	1.8500	89	05.03.01	Mo	11.00%	3.29%	5.72%	0.0114	0.1700	0.21
5.2	call	1.8500	88	04.03.01	Su	10.99%	3.29%	5.72%	0.0113	0.0586	0.21
2.8	call	1.8500	87	03.03.01	Sa	10.99%	3.29%	5.72%	0.0111	0.0312	0.21
1.7	call	1.8500	86	02.03.01	Fr	10.99%	3.29%	5.72%	0.0110	0.0187	0.21
1.2	call	1.8500	85	01.03.01	Th	10.99%	3.29%	5.72%	0.0109	0.0130	0.21
0.8	call	1.8500	84	28.02.01	We	10.98%	3.29%	5.72%	0.0107	0.0086	0.21
0.7	call	1.8500	83	27.02.01	Tu	10.98%	3.29%	5.72%	0.0106	0.0074	0.21
0.6	call	1.8500	82	26.02.01	Mo	10.98%	3.29%	5.72%	0.0104	0.0063	0.20
0.5	call	1.8500	81	25.02.01	Su	10.97%	3.29%	5.72%	0.0103	0.0051	0.20
1.0	call	1.8500	79	23.02.01	Fr	10.97%	3.29%	5.72%	0.0100	0.0100	0.20
0.6	call	1.8500	77	21.02.01	We	10.96%	3.29%	5.72%	0.0097	0.0058	0.20
0.5	call	1.8500	74	18.02.01	Su	10.95%	3.29%	5.72%	0.0093	0.0046	0.19
0.5	call	1.8500	71	15.02.01	Th	10.94%	3.29%	5.72%	0.0088	0.0044	0.19
0.5	call	1.8500	67	11.02.01	Su	10.93%	3.29%	5.72%	0.0082	0.0041	0.18
0.4	call	1.8500	62	06.02.01	Tu	10.90%	3.29%	5.72%	0.0075	0.0030	0.18
0.4	call	1.8500	58	02.02.01	Fr	10.91%	3.29%	5.72%	0.0069	0.0028	0.17
0.3	call	1.8500	52	27.01.01	Sa	10.92%	3.29%	5.72%	0.0060	0.0018	0.16
0.3	call	1.8500	44	19.01.01	Fr	10.94%	3.29%	5.72%	0.0048	0.0014	0.14
0.3	call	1.8500	36	11.01.01	Th	10.97%	3.29%	5.72%	0.0036	0.0011	0.12
0.3	call	1.8500	26	01.01.01	Mo	11.00%	3.29%	5.72%	0.0021	0.0006	0.09
0.2	call	1.8500	11	17.12.00	Su	11.00%	3.29%	5.72%	0.0003	0.0001	0.02

Figure 8: portfolio of call options for a static hedge

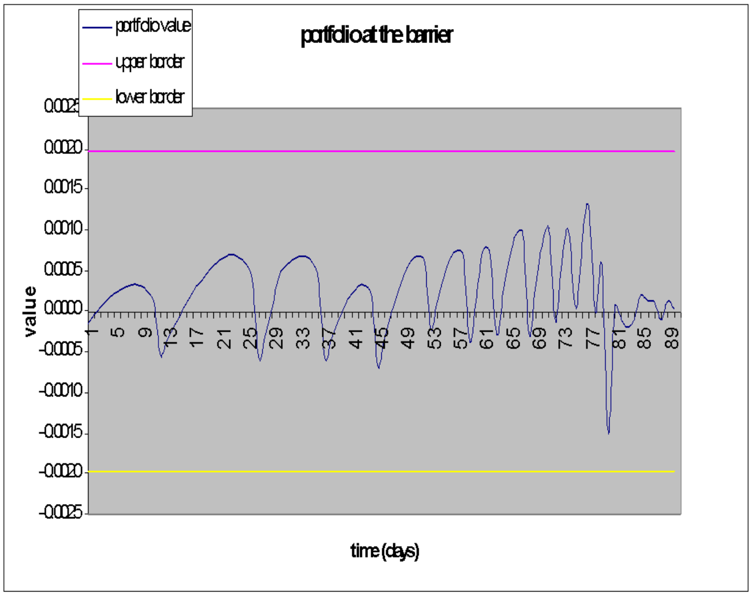


Figure 9: value of the portfolio at the barrier for a static hedge of an up-and-out call using vanilla call options

4.2 Vega hedging

Besides delta hedging the major uncertainty is the volatility. Vega hedging is important. This can be done by trading vanillas. Consider the setup

- 3 months up-and-out put
- spot = 0.94, strike = 1.01, barrier = 0.98
- domestic rate = 3.05%, foreign rate = 6.5%, volatility = 11.0%
- theoretical value = 0.0481.
- consider the curve vega depending on spot.
- determine 1 or 2 vanilla options to offset this vega curve
- by changing strike, maturity and nominal
- in the example: sell 0.9 calls with strike 0.9492 and 90 days maturity and buy 0.8 calls with strike 0.9776 and 60 days maturity

Figures 10 and 11 show the corresponding vega curves after 21 days and after 45 days.

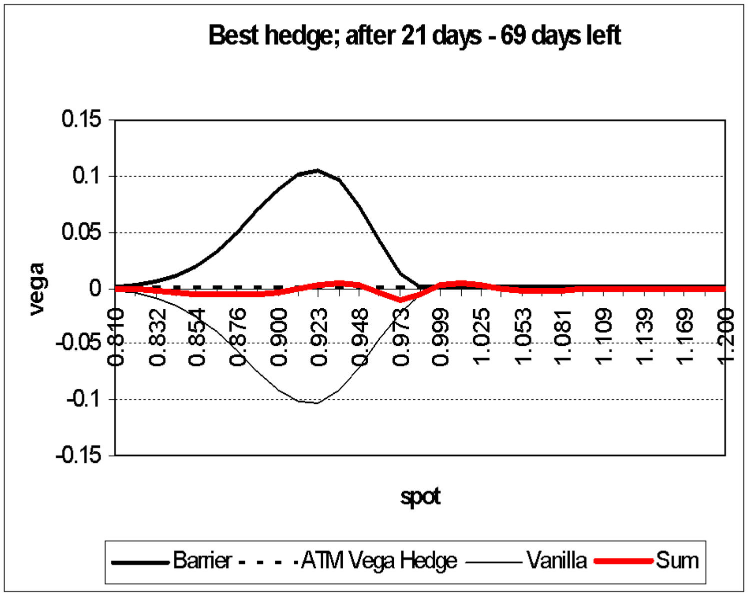


Figure 10: vega depending on spot of an up-and-out put and a vega hedge consisting of two vanilla options

5 Applying dynamic hedging to barrier options

Dynamic Hedging consists of

- investing delta in the underlying
- offset the vega position with suitable vanilla options

Since delta hedging is impractical, we show alternatives in the next section

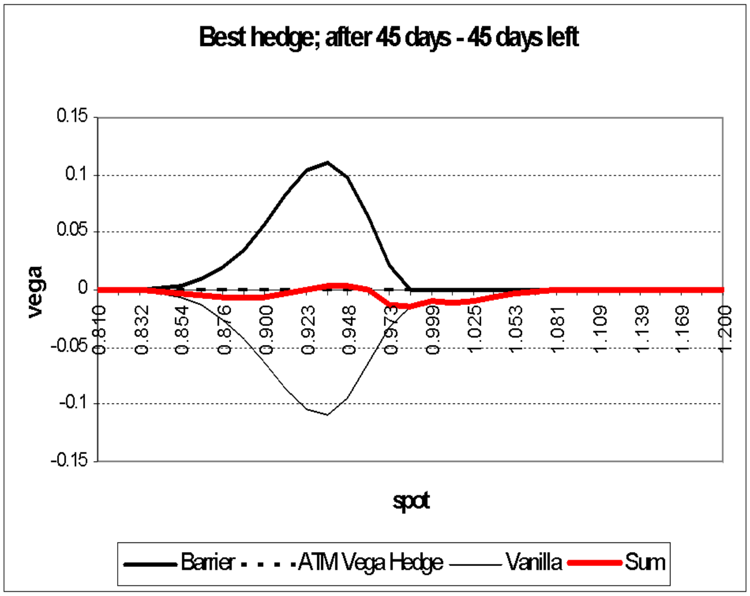


Figure 11: vega depending on spot of an up-and-out put and a vega hedge consisting of two vanilla options

6 Over-hedging strategies

6.1 Constrained portfolios such as limited leverage

6.1.1 Model formulation and survey of super- replication under leverage constraints

- Ω continuous functions from $[0,T]$ to $\mathbb{R}$ taking the value zero at zero,
- $\mathbb{P}$ to be Wiener measure,
- $W(t,\omega) = \omega(t)$ for all $t \in [0,T]$ and all $\omega \in \Omega$.
- $\mathcal{F}^W(t)$: the σ-algebra generated by $(W(s); 0 \le s \le t)$.
- The σ-algebra $\mathcal{F}(T)$ is the $\mathbb{P}$-completion of $\mathcal{F}^W(T)$, and for $0 \le t \le T$, $\mathcal{F}(t)$ is the augmentation of $\mathcal{F}^W(t)$ by the $\mathbb{P}$-null sets of $\mathcal{F}(T)$.
- contingent claim whose payoff at expiration date T is $g(S(\cdot))$.
- $C_+[0,T]$ denote the space of nonnegative continuous functions on $[0,T]$.
- We assume that the nonnegative function $g \colon C_+[0,T] \to [0,\infty)$ is lower semicontinuous. The argument of g is the path of the stock price process S from date 0 to date T, and because this path is random, $g(S(\cdot))$ is a random variable on $(\Omega, \mathcal{F}(T), \mathbb{P})$.

The problem of super-replication of a short position in this option can be posed as follows.

- $X(0) > 0$: given nonrandom *initial wealth,*
- choose an $(\mathcal{F}(t); 0 \le t \le T)$-adapted *portfolio process* $(\pi(t); 0 \le t \le T)$
- and *cumulative consumption process* $(C(t); 0 \le t \le T)$.

- We interpret $\pi(t)$ as the proportion of wealth invested in the stock at time t (sometimes called the *gearing*).
- The remaining wealth is invested at interest rate r,
- $C(t)$ is the amount of wealth consumed up to time t.

This leads us to model the differential of wealth as

$$\begin{aligned} dX(t) &= \pi(t)X(t)\frac{dS(t)}{S(t)} + rX(t)(1-\pi(t))\,dt - dC(t) \\ &= rX(t)\,dt + \sigma\pi(t)X(t)\,dW(t) - dC(t). \end{aligned} \tag{110}$$

If $X(T) \geq g(S(\cdot))$ almost surely (a.s.), we say that (π, C) *super-replicates* $g(S(\cdot))$ beginning with initial wealth $X(0)$.

Next, given some fixed number $\alpha \in [0,\infty)$, we impose the *portfolio constraint*

$$\pi(t) \geq -\alpha, \quad 0 \leq t \leq T, \text{ a.s.} \tag{111}$$

The point of this constraint, in the context of the knock-out call of the previous section, is to avoid short positions which are too large relative to the value of the contingent claim being hedged. The parameter α must be chosen by the person pricing the contingent claim; in the case of the knock-out call, we interpret α in terms of a transaction cost in Section 6.2.12, and this provides a guide to choosing it. If $\alpha = 0$, then short positions in the underlying are prohibited.

The *upper hedging price* of the contingent claim $g(S(\cdot))$ is defined to be

$$\begin{aligned} & v(0, S(0); \alpha) \\ \triangleq\ & \inf\left\{ X(0) \,\middle|\, \begin{array}{l} \text{There exist } \pi \text{ and } C \text{ satisfying (111)} \\ \text{such that } X(T) \geq g(S(\cdot)) \text{ a.s.} \end{array} \right\}. \end{aligned} \tag{112}$$

- Cvitanić & Karatzas [8] have shown that when $v(0, S(0); \alpha)$ is finite, there exists an $X(0)$, denoted $\widehat{X}(0)$, and corresponding portfolio and consumption processes $\hat{\pi}$ and $\widehat{C}$ attaining the infimum in (112).
- We denote the corresponding wealth process by $\widehat{X}(t)$, $0 \leq t \leq T$.
- For $0 \leq t < T$, we define the *upper hedging price at time* t of the contingent claim $g(S(\cdot))$ to be $\widehat{X}(t)$.
- The upper hedging price $\widehat{X}(t)$ generally exceeds the risk-neutral price $\mathbb{E}[e^{-r(T-t)}g(S(\cdot))\,|\,\mathcal{F}(t)]$ because the upper hedging price includes a "reserve" to offset the portfolio constraint. During the evolution of the process, some part of this reserve might be revealed to be unnecessary. The process $\widehat{C}$ is included in the formulation of the upper hedging price so that unnecessary reserve can be removed and thus no longer included in the upper hedging price.

Theorem 6.1 *[Cvitanić & Karatzas, El Karoui & Quenez] The upper hedging price of (112) satisfies*

$$v(0, S(0); \alpha) = \sup_{\lambda} \mathbb{E}_{\lambda}\big[e^{-rT-\alpha\lambda(T)} g(S(\cdot))\big], \tag{113}$$

where the supremum is over all adapted, nondecreasing, processes which are Lipschitz continuous in t, uniformly in ω, and satisfy λ(0) = 0. Here $\mathbb{E}_\lambda$ *denotes expectation under the probability measures* $\mathbb{P}_\lambda$ *whose Radon-Nikodým derivative with respect to* $\mathbb{P}$ *is*

$$\frac{d\mathbb{P}_\lambda}{d\mathbb{P}} = \exp\left\{ -\frac{1}{\sigma}\int_0^T \lambda'(t)\,dW(t) - \frac{1}{2\sigma^2}\int_0^T (\lambda'(t))^2 dt \right\}. \tag{114}$$

The supremum in (113) over Lipschitz continuous processes is often not attained, and Lipschitz continuity is not easily relaxed in Theorem 6.1 because of the need to define $\mathbb{P}_\lambda$ by (114).

Broadie, Cvitanić & Soner [5] specialized Theorem 6.1 to the case of a contingent claim whose payoff at expiration is a function of the final value of a single, geometric Brownian motion. A presentation of the results of both [8] and [5] in full generality may be found in [19].

Theorem 6.2 (Broadie, Cvitanić & Soner)
Let $\varphi\colon [0,\infty) \to [0,\infty)$ be lower semicontinuous, and suppose the contingent claim $g(S(\cdot))$ is given by $g(S(\cdot)) = \varphi(S(T))$. Define the "face-lifted" payoff function

$$\widehat{\varphi}_\alpha(x) \triangleq \sup_{\lambda \geq 0} e^{-\alpha\lambda}\varphi(xe^{-\lambda}), \quad x \geq 0. \tag{115}$$

Then the upper hedging price under hedge-portfolio constraint (111) is given by

$$v(0, S(0); \alpha) = \mathbb{E}\big[e^{-rT}\widehat{\varphi}_\alpha(S(T))\big]. \tag{116}$$

Schmock, Shreve and Wystup [27] formulated the dual problem in such a way that no change of measure is required, and they could then extend the class of processes over which the supremum in the dual problem is computed. Their goal was to extend Theorem 6.2 to path-dependent options. Their main result is that in place of the "face-lifting" procedure (115), one must solve a singular stochastic control problem. This problem can sometimes be solved by inspection, and in particular, such a solution is possible for the knock-out call of the previous section. The solution of the stochastic control problem leads directly to a formula for the upper hedging price, in the spirit of (116).

The work by Schmock, Shreve and Wystup [27] is more general than [5] in that it allows path-dependent options, but more special in that the only portfolio constraint considered there is (111), whereas [5] permits a general convex constraint on π. They converted the computation of the supremum on the right-hand side of (113) to a singular stochastic control problem and proved the following theorem.

Theorem 6.3 (Schmock, Shreve & Wystup) *Let g be a nonnegative, lowersemicontinuous function defined on $C_+[0,T]$. The upper hedging price for the contingent claim with payoff $g(S)$ at expiration date T and hedge-portfolio constraint (111) is*

$$v(0, S(0); \alpha) = \sup_{\lambda \in \mathcal{C}} \mathbb{E}\big[e^{-r_d T - \alpha\lambda(T)} g(Se^{-\lambda})\big], \tag{117}$$

where S denotes the solution of (82)*, namely*

$$S(t) = S(0)\exp\big(\sigma W(t) + rt - \tfrac{1}{2}\sigma^2 t\big), \quad 0 \leq t \leq T \tag{118}$$

and $\mathcal{C}$ is defined by

$$\begin{aligned}\mathcal{C} \triangleq \big\{\lambda;\, &\lambda \textit{ is an } \{\mathcal{F}(t); 0 \leq t \leq T\}\textit{-adapted,}\\ &\textit{nondecreasing, continuous process with } \lambda(0) = 0\big\}.\end{aligned} \tag{119}$$

remarks

- The problem of maximizing $\mathbb{E}\big[e^{-r_d T - \alpha\lambda(T)} g(Se^{-\lambda})\big]$ over all $\lambda \in \mathcal{C}$ is one of stochastic control.
- It turns out that there is often a sequence $\{\lambda_n\}_{n=1}^\infty$ in $\mathcal{C}$ with

  $$\lim_{n\to\infty} \mathbb{E}\big[e^{-r_d T - \alpha\lambda_n(T)} g(Se^{-\lambda_n})\big] = v(0, S(0); \alpha)$$

  and the limit λ of the sequence $\{\lambda_n\}_{n=1}^\infty$ is a singularly continuous process; hence the characterization of the right-hand side of (117) as a singular stochastic control problem.

- However, the limiting λ can fail to obtain the supremum in (117) because g is lower semicontinuous rather than upper semicontinuous; lower semicontinuity is needed for the proof of Theorem 6.3.
- The difference between Theorems 6.1 and 6.3 is that whereas the former requires a maximization over changes of measure, the latter allows one to maximize over processes $\lambda \in \mathcal{C}$, always computing expectations using the same operator $\mathbb{E}$. Of course, one can use the Radon-Nikodým derivative $d\mathbb{P}_\lambda/d\mathbb{P}$ to rewrite the right-hand side of (113) as an expectation under the expectation operator corresponding to $\mathbb{P}$, but the presence of the Radon-Nikodým derivative in the resulting stochastic control problem complicates it considerably.
- As shown in examples, the stochastic control problem of (117) can often be solved by inspection.
- For the examples of Section 6.2.11, it is helpful to generalize Theorem 6.3 to right-continuous λ with possible jumps only at fixed dates.

6.2 Application to reverse knock-out barrier options

6.2.1 Constrained in-the-money knock-out call

For the in-the-money knock-out call the function g is

$$g(y) \triangleq \big(y(T) - K\big)^+ I_{\{\max_{0\le t\le T} y(t) < B\}}, \quad y \in C_+[0,T]. \tag{120}$$

We have chosen to write the set $\{\max_{0\le t\le T} y(t) < B\}$ with the strict inequality so that g will be lower semicontinuous. For geometric Brownian motion (118), the probability of reaching a barrier is the same as the probability of crossing the same barrier, so the contingent claim defined by

$$g^*(y) \triangleq \big(y(T) - K\big)^+ I_{\{\max_{0\le t\le T} y(t) \le B\}}, \quad y \in C_+[0,T], \tag{121}$$

has the same upper hedging price.

We consider the problem of maximization of

$$\mathbb{E}\big[e^{-r_dT-\alpha\lambda(T)}\big(S_\lambda(T) - K\big)^+ I_{\{M_\lambda(T)<B\}}\big], \tag{122}$$

where

$$S_\lambda(t) \triangleq S(t)e^{-\lambda(t)}, \quad M_\lambda(t) \triangleq \max_{0\le u\le t} S_\lambda(u), \tag{123}$$

and $0 < S(0) < B$. The maximization is over processes $\lambda \in \mathcal{C}$. To find the maximal value of (122) it is clear that one should choose the nondecreasing process λ so that $M_\lambda(T)$ is strictly less than B. On the other hand, one should not have λ be any larger than necessary because λ appears in both the discount term $e^{-r_dT-\alpha\lambda(T)}$ and as a discount in the formula for S_λ. If g were given by (121), the maximizing λ causes reflection of S_λ at the barrier B, i.e.,

$$\lambda^*(t) \triangleq \max_{0\le u\le t}\big(\log S(u) - \log B\big)^+. \tag{124}$$

Since g is dominated by g^*, we have

$$v(0, S(0); \alpha) \le \mathbb{E}\big[e^{-r_dT-\alpha\lambda^*(T)}\big(S_{\lambda^*}(T) - K\big)^+\big]. \tag{125}$$

But with g given by (120), we choose a sequence of barriers $\{B_n\}_{n=1}^\infty$ converging up to B but always strictly less than B and then take the sequence of processes $\{\lambda_n\}_{n=1}^\infty$ for which λ_n causes reflection at B_n. Then $\lambda_n(T) \downarrow \lambda^*(T)$ and $S_{\lambda_n}(T) \uparrow S_{\lambda^*}(T)$ as $n \to \infty$. By the bounded convergence theorem,

$$\begin{aligned} v(0, S(0); \alpha) &\ge \limsup_{n\to\infty} \mathbb{E}\big[e^{-r_dT-\alpha\lambda_n(T)}\big(S_{\lambda_n}(T) - K\big)^+\big] \\ &= \mathbb{E}\big[e^{-r_dT-\alpha\lambda^*(T)}\big(S_{\lambda^*}(T) - K\big)^+\big]. \end{aligned} \tag{126}$$

These considerations have led us to the following corollary of Theorem 6.3.

Corollary 6.4 *For $0 \leq t \leq T$ and $0 < x \leq B$, define*

$$\begin{aligned} & v^*(t,x;\alpha) && (127)\\ \triangleq\; & \mathbb{E}\big[e^{-r_d(T-t)-\alpha(\lambda^*(T)-\lambda^*(t))}\big(S_{\lambda^*}(T)-K\big)^+ \mid S_{\lambda^*}(t)=x\big]. \end{aligned}$$

Let $t \in [0,T]$ be given, and assume that $S(t)=x$. Then the upper hedging price at time t of the in-the-money knock-out call is

$$v(t,x;\alpha) = v^*(t,x;\alpha) I_{\{x<B\}}, \tag{128}$$

and for $t \in [0,T)$ the function $v^*(t,x;\alpha)$ can be computed explicitly (see Section 6.2.2 for details).

furthermore, we have

$$\alpha v^*(t,B;\alpha) + B v_x^*(t,B;\alpha) = 0, \tag{129}$$

$$v_t^*(t,x;\alpha) + rx v_x^*(t,x;\alpha) + \frac{1}{2}\sigma^2 x^2 v_{xx}^*(t,x;\alpha) = r_d v^*(t,x;\alpha). \tag{130}$$

6.2.2 Analytical Solutions

analytical solutions for

digital options

reverse barrier options

one-touch digital options

To model leverage constraints we will always take

- $\pi \leq \alpha$ for borrowing constraints
- $\pi \geq -\alpha$ for short-selling constraints ($\alpha \geq 0$)

The first example of digital options is a straightforward application of face-lifting Theorem 6.2 (Broadie, Cvitanić & Soner). Unfortunately, this type of face-lifting does not work for path-dependent options. Therefore, reverse barrier options and one-touch digital options require separate treatment, which is presented in the sequel.

6.2.3 Digital Options

Digital options, also called binary options, have the path-independent payoff

$$g(S) = \varphi(S_T) = I_{\{\phi S_T \geq \phi B\}}, \tag{131}$$

where B denotes the strike and ϕ is a binary variable taking the values $\phi=+1$ for a digital call and $\phi=-1$ for a digital put. We impose the natural constraint

$$\alpha v - \phi x v_x \geq 0 \tag{132}$$

for some $\alpha \geq 0$ and compute the constrained value function $v(t,x;\alpha)$ by the face-lifting method

$$v(0,S_0;\alpha) = e^{-r_d T}\mathbb{E}[\hat{\varphi}(S_T)]. \tag{133}$$

We obtain

$$\hat{\varphi}(S_T) = I_{\{\phi S_T \geq \phi B\}} + \left(\frac{S_T}{B}\right)^{\phi\alpha} I_{\{\phi S_T < \phi B\}} \tag{134}$$

and evaluate $v(0, S_0; \alpha)$

$$\begin{aligned}
&= e^{-r_d T}\left[\int_{\{\phi S_0 e^{\sigma\sqrt{T}x+\sigma\sqrt{T}\theta_-}\geq\phi B\}} \mathcal{N}'(x)\,dx \right.\\
&\quad \left. + \int_{\{\phi S_0 e^{\sigma\sqrt{T}x+\sigma\sqrt{T}\theta_-}<\phi B\}} \left(\frac{S_0 e^{\sigma\sqrt{T}x+\sigma\sqrt{T}\theta_-}}{B}\right)^{\phi\alpha} \mathcal{N}'(x)\,dx\right] \qquad (135)\\
&= e^{-r_d T}\left[\mathcal{N}(\phi d_-) + \left(\frac{S_0}{B}\right)^{\phi\alpha} e^{\phi\alpha\theta_-\sigma\sqrt{T}+\frac{1}{2}\alpha^2\sigma^2 T}\mathcal{N}(-\phi d_- - \alpha\sigma\sqrt{T})\right],
\end{aligned}$$

where we denote

$$d_\pm \triangleq \frac{\ln\frac{S_0}{B}}{\sigma\sqrt{T}} + \theta_\pm, \qquad (136)$$

$$\theta_\pm \triangleq \left(\frac{r_d - r_f}{\sigma} \pm \frac{\sigma}{2}\right)\sqrt{T}. \qquad (137)$$

We obtain the constrained value function $v(t, x; \alpha)$ by replacing S_0 by x, and T by $T - t \triangleq \tau$. The *danger-supplement* can be read off as

$$h(t, x; \alpha) = e^{-r_d\tau}\left(\frac{x}{B}\right)^{\phi\alpha} e^{\phi\alpha\theta_-\sigma\sqrt{\tau}+\frac{1}{2}\alpha^2\sigma^2\tau}\mathcal{N}(-\phi d_- - \alpha\sigma\sqrt{\tau}). \qquad (138)$$

6.2.4 Reverse Barriers

6.2.5 The Up-and-Out Call

We will now price an up-and-out call option subject to the short-selling constraint

$$\pi(t) \geq -\alpha \quad \forall\, t \in [0, T] \qquad (139)$$

for some number $\alpha \geq 0$, which can be written as

$$\alpha v(t, x; \alpha) + x v_x(t, x; \alpha) \geq 0,\ 0 \leq t \leq T,\ 0 \leq x \leq B. \qquad (140)$$

relation between the short-selling constraint and the face-lifting equation on an intuitive level

- Given a path-independent payoff $g(S) = \varphi(S_T)$, we want to compute the face-lifted $\hat{\varphi}$ as defined in equation (115).
- To do that, we need to maximize the real function $f(\nu) \triangleq \varphi(xe^{-\nu})\,e^{-\alpha\nu}$ for each value of x.
- Denoting $y \triangleq xe^{-\nu}$, the first order condition $f'(\nu) = 0$ implies

$$\alpha\varphi(y) + y\varphi'(y) = 0. \qquad (141)$$

- We see that the short-selling constraint is imposed *with equality at the* final *boundary* of the region where $v(t, x; \alpha)$ must satisfy the Black-Scholes differential equation.
- One can check that if $v(t, x)$ satisfies the Black-Scholes equation, the function $\alpha v(t, x) + x v_x(t, x)$ also satisfies the Black-Scholes equation (assuming enough differentiability).
- It is now a consequence of the *maximum-principle* that the constraint holds inside this region as well, but not necessarily with equality.

- The reason why the short-selling constraint is imposed with equality at the final time is to get the minimality of the value function.

This intuition leads to the following idea to determine the constrained value function $v(t,x;\alpha)$ of the up-and-out call option. We impose the short-selling constraint *with equality on the boundary* of the region where the option is defined and where the unconstrained value function violates the short-selling constraint, i.e., instead of solving the partial differential equation setup for $v(t,x)$,

$$\begin{aligned}
\mathcal{L}v &= 0 \ \forall t \in [0,T), x \in (0,B), && (142)\\
v(t,x) &= 0 \ \forall t \in [0,T], x > B, && (143)\\
v(t,B) &= 0 \ \forall t \in [0,T], && (144)\\
v(t,0) &= 0 \ \forall t \in [0,T], && (145)\\
v(T,x) &= (x-K)^+ I_{\{x<B\}} \ \forall x \geq 0, && (146)
\end{aligned}$$

where the Black-Scholes differential operator is defined by

$$\mathcal{L}v \triangleq -r_d v + v_t + (r_d - r_f) x v_x + \frac{1}{2}\sigma^2 x^2 v_{xx}, \tag{147}$$

we seek a function $v(t,x;\alpha)$ satisfying

$$\begin{aligned}
\mathcal{L}v &= 0 \ \forall t \in [0,T), x \in (0,B), && (148)\\
v(t,x;\alpha) &= 0 \ \forall t \in [0,T], x > B, && (149)\\
\alpha v(t,B;\alpha) + B v_x(t,B;\alpha) &= 0 \ \forall t \in [0,T], && (150)\\
v(t,0;\alpha) &= 0 \ \forall t \in [0,T], && (151)\\
v(T,x;\alpha) &= (x-K)^+ I_{\{x\leq B\}} \ \forall x \geq 0 && (152)
\end{aligned}$$

and claim that the solution is the upper hedging price of the constrained up-and-out call at time t if $S(t) = x < B$.

To see this we first set

$$M(t) \triangleq \max_{0\leq u\leq t} S(u). \tag{153}$$

Next we define the value of an *auxiliary contingent claim* by

$$\begin{aligned}
& w(t,x;\alpha) && (154)\\
\triangleq \ & \mathbb{E}\left[e^{-r_d(T-t)}[(1+\alpha)S(T) - \alpha K] I_{\{S(T)\geq K\}} I_{\{M(T)<B\}} \middle| S_t = x \right].
\end{aligned}$$

The method for finding auxiliary value functions is to consider the function

$$\alpha v(t,x;\alpha) + x v_x(t,x;\alpha). \tag{155}$$

We would like to *define* this to be the auxiliary value function $w(t,x;\alpha)$. The problem is that $v(t,x;\alpha)$ is yet to be computed, whence we cannot use it to define $w(t,x;\alpha)$. Instead, we use the desired identity

$$w(t,x;\alpha) = \alpha v(t,x;\alpha) + x v_x(t,x;\alpha) \tag{156}$$

only to compute terminal and boundary conditions for w and try to identify the auxiliary contingent claim w. *Then* we solve for each t the *ordinary* differential equation (156) to obtain a *candidate* for $v(t,x;\alpha)$ in terms of $w(t,x;\alpha)$. Finally we have to verify whether the value function $v(t,x;\alpha)$ has all the properties we want.

Details. Define the first hitting time τ by

$$\tau \triangleq T \wedge \inf\{t : S(t) = B\} \tag{157}$$

We list some properties of $w(t, x; \alpha)$ which follow immediately from its definition.

(i) $\left\{e^{-r_d(t\wedge\tau)}w(t \wedge \tau, S(t \wedge \tau); \alpha)\right\}_t$ is a martingale, and therefore

(ii) $w(t, x; \alpha)$ satisfies the Black-Scholes partial differential equation for $t \in [0, T)$ and $x \in (0, B)$.

(iii) $w(t, B; \alpha) = 0$ for $t \in [0, T]$.

(iv) $0 \le w(t, x; \alpha) \le (1 + \alpha)x$ for $t \in [0, T]$ and $x \in [0, B]$ and thus $w(t, 0; \alpha) = 0$ for $t \in [0, T]$.

(v) $w(T, x; \alpha) = [(1 + \alpha)x - \alpha K]I_{\{x\ge K\}}I_{\{x<B\}}$ for $x \in [0, B]$.

(vi) $w(t, x; \alpha)$ is continuous and nonnegative for $(t, x) \in [0, T] \times [0, B]$.

(vii) $w(t, x; \alpha) = 0$ for $x \ge B$ and $t \in [0, T]$.

Now we can define

$$v(t, x; \alpha) \triangleq \int_0^1 y^{\alpha-1}w(t, xy; \alpha)\, dy = x^{-\alpha} \int_0^x z^{\alpha-1}w(t, z; \alpha)\, dz \tag{158}$$

and list properties of $v(t, x; \alpha)$ which follow from its definition:

(i) $v(t, x; \alpha)$ satisfies the Black-Scholes partial differential equation for $t \in [0, T)$ and $x \in (0, B)$.

(ii) $0 \le v(t, x; \alpha) \le x$ for $t \in [0, T]$ and $x \in [0, B]$ and thus $v(t, 0; \alpha) = 0$ for $t \in [0, T]$.

(iii) $xv_x(t, x; \alpha) + \alpha v(t, x; \alpha) = w(t, x; \alpha)$ for $t \in [0, T)$ and $x \in (0, B)$ and therefore by continuity

(iv) $Bv_x(t, B; \alpha) + \alpha v(t, B; \alpha) = 0$ for $t \in [0, T)$. We mean by $v_x(t, B; \alpha)$ the quantity

$$\lim_{h\downarrow 0} \frac{v(t, B; \alpha) - v(t, B - h; \alpha)}{h},$$

because we would like to leave the value of $v(t, x; \alpha)$ unspecified when $x > B$.

(v) $v(T, x; \alpha) = (x - K)^+$ for $x \in [0, B]$.

(vi) $v(t, B; \alpha) = \int_0^1 y^{\alpha-1}w(t, By; \alpha)\, dy$ for $t \in [0, T]$.

(vii) $v(t, x; \alpha)$ is continuous for $(t, x) \in [0, T] \times [0, B]$.

(viii) $\lim_{x\to 0} xv_x(t, x) = 0$ for $t \in [0, T]$.

(ix) $v(t, x; \alpha) > v(t, x; \infty)$ (follows from the maximum principle).

(x) $\lim_{\alpha\to\infty} v(t, x; \alpha) = v(t, x)$, as we would expect.

- $v(t, x; \alpha)$ solves the Black-Scholes partial differential equation (148) subject to the boundary conditions (150) and (151) and the terminal condition (152),
- and in addition $\pi(t, x) = \frac{xv_x(t,x;\alpha)}{v(t,x;\alpha)}$ super-replicates the payoff of an up-and-out call option and satisfies the short-selling constraint during its lifetime.

We will now demonstrate that the function $v(t,x;\alpha)$ derived above is the smallest function which super-replicates the payoff of an up-and-out call and satisfies the Black-Scholes partial differential equation and the short-selling constraint, which will complete the argument that $v(t,x;\alpha)$ is the upper hedging price.

To do this, we show that any other function $\tilde{v}(t,x;\alpha)$, which satisfies

- the Black-Scholes partial differential equation for $t \in [0,T)$ and $x \in (0,B)$,
- $\tilde{v}(T,x;\alpha) = v(T,x;\alpha)$ for $x \in [0,B]$,
- and the constraint $\alpha\tilde{v}(t,x;\alpha) + x\tilde{v}_x(t,x;\alpha) \geq 0$ for $t \in [0,T)$ and $x \in [0,B]$, where we take one-sided derivatives at the endpoints 0 and B,

cannot be less than $v(t,x;\alpha)$. Since $\tilde{v}$ also satisfies the short-selling constraint at the barrier, but perhaps not with equality, let

$$\alpha\tilde{v}(t,B;\alpha) + B\tilde{v}_x(t,B;\alpha) \triangleq g(t),\ 0 \leq t \leq T, \tag{159}$$

for some nonnegative function g. Then $\tilde{v}$ can be characterized in the same way as v, namely by defining

$$\tilde{v}(t,x;\alpha) \triangleq \int_0^1 y^{\alpha-1}\tilde{w}(t,xy;\alpha)dy \tag{160}$$

where

(i) $\tilde{w}(t,x;\alpha)$ satisfies the Black-Scholes partial differential equation for $t \in [0,T)$ and $x \in (0,B)$,

(ii) $\tilde{w}(T,x;\alpha) = w(T,x;\alpha)$ for $x \in [0,B]$,

(iii) $\tilde{w}(t,0;\alpha) = w(t,0;\alpha) = 0$ for $t \in [0,T]$,

(iv) $\tilde{w}(t,B;\alpha) = g(t) \geq 0 = w(t,B;\alpha)$ for $t \in [0,T]$.

As before we conclude that

(i) $\tilde{v}(t,x;\alpha)$ satisfies the Black-Scholes partial differential equation for $t \in [0,T)$ and $x \in (0,B)$,

(ii) $\tilde{v}(T,x;\alpha) = v(T,x;\alpha)$ for $x \in [0,B]$,

(iii) $\tilde{v}(t,0;\alpha) = v(t,0;\alpha) = 0$ for $t \in [0,T]$,

(iv) $\alpha\tilde{v}(t,x;\alpha) + x\tilde{v}_x(t,x;\alpha) = \tilde{w}(t,x;\alpha)$ for $t \in [0,T)$ and $x \in [0,B]$ and hence

(v) $\alpha\tilde{v}(t,B;\alpha) + B\tilde{v}_x(t,B;\alpha) = \tilde{w}(t,B;\alpha) = g(t)$ for $t \in [0,T)$.

Since by the maximum principle, $\tilde{w} \geq w$, we can deduce

$$\begin{aligned}\tilde{v}(t,x,\alpha) &= \int_0^1 y^{\alpha-1}\tilde{w}(t,xy;\alpha)dy \\ &\geq \int_0^1 y^{\alpha-1}w(t,xy;\alpha)dy \\ &= v(t,x;\alpha).\end{aligned} \tag{161}$$

Notice that $\tilde{w}$ can be viewed as an auxiliary up-and-out option with rebate $g(t)$, whereas w does not have a rebate. The option with the rebate must be worth at least as much as the option without the rebate. This is the maximum principle in terms of finance. We conclude that $v(t,x;\alpha)$ is the upper hedging price for $x < B$. For $x = B$ the upper hedging price is clearly zero, because the option is knocked out.

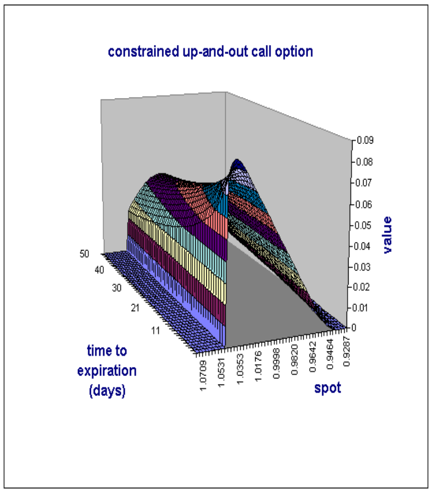


Figure 12: value of a constrained up-and-out call option $v(t, x, \alpha)$ given by (180) with strike $K = 0.95$, knock-out barrier $B = 1.05$ and maturity $T = 90/365$. We used the interest rates $r_d = 5\%$, $r_f = 0\%$, volatility $\sigma = 10\%$ and $\alpha = 50$

compute $v(t, x; \alpha)$ explicitly We need w first. By definition, we know that $w(t, x; \alpha) = 0$ for $x \geq B$. To find $w(t, x; \alpha)$ for $x < B$, we use the joint density (99) for the random pair of a final time value and the running maximum of a Brownian motion with drift and compute the expected value as an integral:

$$\begin{aligned} & w(0, S_0) \\ = \quad & e^{-r_d T} \int_b^m \int_{0 \vee x}^m [(1+\alpha) S_0 e^{\sigma x} - \alpha K] f(x, y) \, dy \, dx \\ = \quad & (1+\alpha) S_0 \left[\mathcal{N}(m - \theta_+) - \mathcal{N}(b - \theta_+)\right] \\ + \quad & (1+\alpha) S_0 e^{2m\theta_+} \left[\mathcal{N}(m + \theta_+) - \mathcal{N}(2m - b + \theta_+)\right] \\ - \quad & \alpha K e^{-r_d T} \left[\mathcal{N}(m - \theta_-) - \mathcal{N}(b - \theta_-)\right] \\ - \quad & \alpha K e^{-r_d T} e^{2m\theta_-} \left[\mathcal{N}(m + \theta_-) - \mathcal{N}(2m - b + \theta_-)\right]. \end{aligned} \tag{162}$$

Here we abbreviate $m \triangleq \frac{1}{\sigma\sqrt{T}} \log \frac{B}{S_0}$, $b \triangleq \frac{1}{\sigma\sqrt{T}} \log \frac{K}{S_0}$ and $\theta_\pm = (\frac{r_d - r_f}{\sigma} \pm \frac{\sigma}{2})\sqrt{T}$.

Finally, it turns out that the integration of definition (158) needed to find the constrained value function v can be performed as well, and the result is given in equation (180). See Figure 12 for a graph of $v(t, x; \alpha)$.

6.2.6 Joint Formulae for all Reverse Barriers

There are four types of reverse barrier options:

1. $(\phi = 1, \eta = 1)$: the down-and-out call,
2. $(\phi = 1, \eta = -1)$: the up-and-out call,

3. ($\phi = -1, \eta = 1$): the down-and-out put,
4. ($\phi = -1, \eta = -1$): the up-and-out put.

Since their analysis is similar to the up-and-out call, we just list the results covering all four types. The suiting constraints are $\pi \geq -\alpha$ for $\eta = -1$ and $\pi \leq \alpha$ for $\eta = 1$. The auxiliary value function $w(t, x; \alpha)$ satisfying the Black-Scholes partial differential equation, the boundary condition $w(t, B; \alpha) = 0$ and the terminal condition

1. down-and-out call:
$$w(T, x; \alpha) = [(\alpha - 1)x - \alpha K] I_{\{x \geq (\frac{\alpha}{\alpha-1}K) \vee B\}} \tag{163}$$
2. up-and-out call:
$$w(T, x; \alpha) = [(\alpha + 1)x - \alpha K] I_{\{K \leq x < B\}} \tag{164}$$
3. down-and-out put:
$$w(T, x; \alpha) = [\alpha K - (\alpha - 1)x] I_{\{B < x \leq K\}} \tag{165}$$
4. up-and-out put:
$$w(T, x; \alpha) = [\alpha K - (\alpha + 1)x] I_{\{x \leq (\frac{\alpha}{\alpha+1}K) \wedge B\}}. \tag{166}$$

The solution is $w(t, x; \alpha)$

$$\begin{aligned}
&= (\alpha - \eta) x e^{-r_f \tau} \left[\frac{\phi - \eta}{2} \mathcal{N}(\phi(m - \theta_+)) + \eta \mathcal{N}(-\eta(b - \theta_+)) \right] \\
&+ (\alpha - \eta) x e^{-r_f \tau} e^{2m\theta_+} \left[\frac{\phi - \eta}{2} \mathcal{N}(\phi(m + \theta_+)) - \phi \mathcal{N}(\phi(l + \theta_+)) \right] \\
&- \alpha K e^{-r_d \tau} \left[\frac{\phi - \eta}{2} \mathcal{N}(\phi(m - \theta_-)) + \eta \mathcal{N}(-\eta(b - \theta_-)) \right] \\
&- \alpha K e^{-r_d \tau} e^{2m\theta_-} \left[\frac{\phi - \eta}{2} \mathcal{N}(\phi(m + \theta_-)) - \phi \mathcal{N}(\phi(l + \theta_-)) \right],
\end{aligned} \tag{167}$$

where

$$\tau = T - t, \tag{168}$$
$$m = \frac{1}{\sigma\sqrt{\tau}} \ln\left(\frac{B}{x}\right), \tag{169}$$
$$b = \left\{ \begin{array}{ll} \frac{1}{\sigma\sqrt{\tau}} \ln\left(\frac{K}{x}\right) & \text{if } \phi\eta = -1 \\ \frac{1}{\sigma\sqrt{\tau}} \ln\left(\frac{\eta \max[\eta B, \frac{\eta\alpha}{\alpha-\eta}K]}{x}\right) & \text{if } \phi\eta = +1 \end{array} \right\} \tag{170}$$
$$l = 2m - b. \tag{171}$$

The constrained value function $v(t, x; \alpha)$ is defined by

$$v(t, x; \alpha) \triangleq \int_0^1 y^{\alpha-1} w(t, x y^{-\eta}; \alpha) \, dy, \tag{172}$$

satisfies the relation

$$w(t, x; \alpha) = \alpha v(t, x; \alpha) - \eta x v_x(t, x; \alpha) \tag{173}$$

and the terminal condition

1. down-and-out call:

$$K' = \frac{\alpha}{\alpha-1}K \tag{174}$$

$$v(T,x;\alpha) = \left\{ \begin{array}{ll} x-K & \text{if } x \geq K' \vee B \\ (K'-K)\left(\frac{x}{K'}\right)^{\alpha} & \text{if } B \leq x \leq K' \\ 0 & \text{if } x < B \end{array} \right\} \tag{175}$$

2. up-and-out call:

$$v(T,x;\alpha) = [x-K]^{+} I_{\{x \leq B\}} \tag{176}$$

3. down-and-out put:

$$v(T,x;\alpha) = [K-x]^{+} I_{\{x \geq B\}} \tag{177}$$

4. up-and-out put:

$$K' = \frac{\alpha}{\alpha+1}K \tag{178}$$

$$v(T,x;\alpha) = \left\{ \begin{array}{ll} K-x & \text{if } x \leq K' \wedge B \\ (K-K')\left(\frac{K'}{x}\right)^{\alpha} & \text{if } K' \leq x \leq B \\ 0 & \text{if } x > B \end{array} \right\} \tag{179}$$

and can be summarized as ($s = (1-\eta\alpha)\sigma\sqrt{\tau}, \quad \tilde{s} = -\eta\alpha\sigma\sqrt{\tau}$)

$$\begin{aligned} v(t,x;\alpha) = xe^{-r_f\tau} & \left[\frac{\phi-\eta}{2}\mathcal{N}\left(-\eta(m-\theta_+)\right) + \eta\mathcal{N}\left(-\eta(b-\theta_+)\right) + e^{\frac{1}{2}s\tau(s-2\theta_+)}\right. \\ & \left.\left\{\frac{\phi-\eta}{2}e^{sm}\mathcal{N}\left(-\eta(-m+\theta_+-s)\right) + \eta e^{sb}\mathcal{N}\left(-\eta(-b+\theta_+-s)\right)\right\}\right] \\ +xe^{-r_f\tau+2m\theta_+}\frac{s}{s-2\theta_+} & \left[\frac{\phi-\eta}{2}\mathcal{N}\left(-\eta(m+\theta_+)\right) - \phi\mathcal{N}\left(\phi(l+\theta_+)\right) + e^{\frac{1}{2}s\tau(s-2\theta_+)}\right. \\ & \left.\left\{\frac{\phi-\eta}{2}e^{(s-2\theta_+)m}\mathcal{N}\left(-\eta(-m+\theta_+-s)\right) + \eta e^{(s-2\theta_+)l}\mathcal{N}\left(-\eta(-l+\theta_+-s)\right)\right\}\right] \\ -Ke^{-r_d\tau} & \left[\frac{\phi-\eta}{2}\mathcal{N}\left(-\eta(m-\theta_-)\right) + \eta\mathcal{N}\left(-\eta(b-\theta_-)\right) + e^{\frac{1}{2}\tilde{s}\tau(\tilde{s}-2\theta_-)}\right. \\ & \left.\left\{\frac{\phi-\eta}{2}e^{\tilde{s}m}\mathcal{N}\left(-\eta(-m+\theta_--\tilde{s})\right) + \eta e^{\tilde{s}b}\mathcal{N}\left(-\eta(-b+\theta_--\tilde{s})\right)\right\}\right] \\ -Ke^{-r_d\tau}e^{2m\theta_-}\frac{\tilde{s}}{\tilde{s}-2\theta_-} & \left[\frac{\phi-\eta}{2}\mathcal{N}\left(-\eta(m+\theta_-)\right) - \phi\mathcal{N}\left(\phi(l+\theta_-)\right) + e^{\frac{1}{2}\tilde{s}\tau(\tilde{s}-2\theta_-)}\right. \\ & \left.\left\{\frac{\phi-\eta}{2}e^{(\tilde{s}-2\theta_-)m}\mathcal{N}\left(-\eta(-m+\theta_--\tilde{s})\right) + \eta e^{(\tilde{s}-2\theta_-)l}\mathcal{N}\left(-\eta(-l+\theta_--\tilde{s})\right)\right\}\right]. \end{aligned} \tag{180}$$

Notice that in the second and in the fourth summand the denominator $s - 2\theta_+$ or $\tilde{s} - 2\theta_-$ could be zero for $\alpha = \frac{2(r_d-r_f)}{\sigma^2}$ or $\alpha = \frac{2(r_d-r_f)}{\sigma^2} - 1$ respectively. However, these are both removable discontinuities, and in fact one can apply l'Hôpital's rule to find the correct equation for these two points. We do not state the explicit result here, because it is not more illuminating than the above formula. Since a minor change in α can avoid hitting the two removable discontinuities, this does not cause any problems in practice.

6.2.7 Comparative Statics

We use the already known auxiliary claim w and obtain

$$v_x = -\frac{\eta}{x}(w - \alpha v) \tag{181}$$

$$v_{xx} = -\frac{\eta}{x^2}[x w_x + (\alpha\eta - 1)w + \alpha(1 - \alpha\eta)v] \tag{182}$$

$$v_t = -\eta\frac{\sigma^2}{2} x w_x + \beta w + (r_d - \alpha\beta)v, \tag{183}$$

where we denote

$$\beta \triangleq -\eta[\frac{\sigma^2}{2}(1 - \eta\alpha) + r_f - r_d]. \tag{184}$$

The sensitivity *theta* is most easily obtained via the Black-Scholes partial differential equation. The leverage is given by $\eta(\alpha - \frac{w}{v})$. See Figures 13 and 14 for delta and gamma of the constrained value function of an up-and-out call option.

The sensitivity *delta* of the auxiliary value function $w(t, x; \alpha)$ can be derived as

$$\begin{aligned}
w_x(t,x;\alpha) = (\alpha - \eta)e^{-r_f\tau}&\left[\frac{\phi - \eta}{2}\mathcal{N}(\phi(m - \theta_+)) + \eta\mathcal{N}(-\eta(b - \theta_+))\right] \\
&+ \frac{\alpha - \eta}{\sigma\sqrt{\tau}}e^{-r_f\tau}\left[-\phi\frac{\phi - \eta}{2}\mathcal{N}'(m - \theta_+) + \mathcal{N}'(b - \theta_+)\right] \\
+(\alpha - \eta)e^{2m\theta_+}e^{-r_f\tau}\left(1 - \frac{2\theta_+}{\sigma\sqrt{\tau}}\right)&\left[\frac{\phi - \eta}{2}\mathcal{N}(\phi(m + \theta_+)) - \phi\mathcal{N}(\phi(l + \theta_+))\right] \\
&+ \frac{(\alpha - \eta)e^{2m\theta_+}e^{-r_f\tau}}{\sigma\sqrt{\tau}}\left[-\phi\frac{\phi - \eta}{2}\mathcal{N}'(m + \theta_+) + \mathcal{N}'(l + \theta_+)\right] \\
&- \frac{\alpha K e^{-r_d\tau}}{x\sigma\sqrt{\tau}}\left[-\phi\frac{\phi - \eta}{2}\mathcal{N}'(m - \theta_-) + \mathcal{N}'(b - \theta_-)\right] \\
&- \frac{-2\alpha\theta_- K e^{-r_d\tau}e^{2m\theta_-}}{x\sigma}\left[\frac{\phi - \eta}{2}\mathcal{N}(\phi(m + \theta_-)) - \phi\mathcal{N}(\phi(l + \theta_-))\right] \\
&- \frac{\alpha K e^{-r_d\tau}e^{2m\theta_-}}{x\sigma\sqrt{\tau}}\left[-\phi\frac{\phi - \eta}{2}\mathcal{N}'(m + \theta_-) + \mathcal{N}'(l + \theta_-)\right],
\end{aligned} \tag{185}$$

where τ, m, b, l, $\theta_\pm$ are defined in equations (168), (169), (170), (171) and (109).

These and other formulae are listed in the section “Dangerous Digitals” at `http://www.mathfinance.de`, and an online calculator there computes leverage constrained prices of reverse barrier options.

6.2.8 One-Touch Digitals

Given a hit-level or barrier B, there are two kinds of one-touch digital options, also called American digitals or hit options. In the first (second) kind the holder of the option receives an amount R, if the underlying hits the barrier B during the life time of the option from below (above). We define the binary variables η and ω to be

1. $\eta = -1$, if B is hit from below,
2. $\eta = +1$, if B is hit from above,
3. $\omega = 1$, if R is paid at expiration time T,
4. $\omega = 0$, if R is paid the first time the underlying hits B.

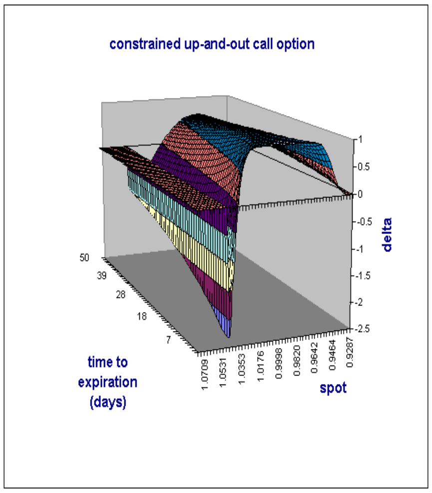


Figure 13: delta of a constrained up-and-out call option $v(t, x, \alpha)$ given by (181) with strike $K = 0.95$, knock-out barrier $B = 1.05$ and maturity $T = 90/365$. We used the interest rates $r_d = 5\%$, $r_f = 0\%$, volatility $\sigma = 10\%$ and $\alpha = 50$

In the case $\eta = -1$ we would want to impose the leverage constraint $\pi \leq \alpha$ and in the case $\eta = +1$ we impose $\pi \geq -\alpha$ for some real number $\alpha \geq 0$ and then find the upper hedging price. As before let us denote by $v(t, x)$ the unconstrained value of the option at time t when the stock price is x and by $v(t, x; \alpha)$ the corresponding constrained value function. Since raising the option value at the boundary, where $v(t, B) = R \exp(-\omega(T - t))$, would make the hedging problem worse, our only chance is to keep the boundary condition $v(t, B; \alpha) = R \exp(-\omega(T - t))$ as it is and increase the terminal condition $v(T, x) = 0$ in a minimal way such that $-\eta\pi \leq \alpha$ holds. This problem has already been solved for the path-independent digital options.

We must choose

$$v(T, x; \alpha) = R \left(\frac{B}{x}\right)^{\eta\alpha}, \quad \eta x \geq \eta B. \tag{186}$$

To solve for $v(t, x; \alpha)$, we observe that it can be decomposed into the sum of the original hit option $v(t, x)$ plus a supplemental *power barrier option* $h(t, x; \alpha)$ defined by

1. the boundary condition $h(t, B; \alpha) = 0$,
2. the terminal condition $h(T, x; \alpha) = R \left(\frac{B}{x}\right)^{\eta\alpha}, \quad \eta x \geq \eta B$,
3. $-r_d h + h_t + (r_d - r_f) x h_x + \frac{1}{2}\sigma^2 x^2 h_{xx} = 0$.

We present the detailed solution for the case $\eta = -1$ following the standard procedure to compute barrier option values. We use the joint probability density function $f(x, y)$ as in equation (99) and compute the present value of the payoff random variable

$$R \left(\frac{S_T}{B}\right)^{\alpha} I_{\{\sup_{0 \leq u \leq T} S_u < B\}} \tag{187}$$

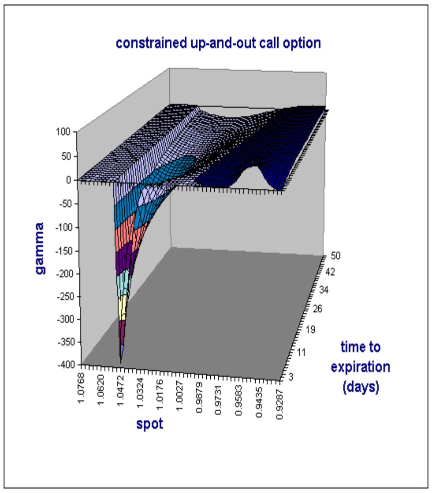


Figure 14: gamma of a constrained up-and-out call option $v(t,x,\alpha)$ given by (182) with strike $K = 0.95$, knock-out barrier $B = 1.05$ and maturity $T = 90/365$. We used the interest rates $r_d = 5\%$, $r_f = 0\%$, volatility $\sigma = 10\%$ and $\alpha = 50$

as

$$\begin{aligned} & \mathbb{E}\left[e^{-r_d T} R \left(\frac{S_T}{B}\right)^{\alpha} I_{\{\sup_{0 \le u \le T} S_u < B\}}\right] \qquad (188) \\ = \; & \frac{Re^{-r_d T}}{B^{\alpha}} \int_{x=-\infty}^{x=m} \int_{y=0 \vee x}^{y=m} (S_0 e^{\sigma x})^{\alpha} f(x,y)\, dy\, dx \\ = \; & Re^{(-r_d + \frac{1}{2}\alpha^2\sigma^2)T + \alpha\sigma\sqrt{T}\theta_-} \\ & \left\{e^{-m\alpha\sigma\sqrt{T}} \mathcal{N}\left(m - \alpha\sigma\sqrt{T} - \theta_-\right)\right. \\ & \left. - e^{m\alpha\sigma\sqrt{T} + 2m\theta_-} \mathcal{N}\left(-m - \alpha\sigma\sqrt{T} - \theta_-\right)\right\}, \end{aligned}$$

where we abbreviate $m \triangleq \frac{1}{\sigma\sqrt{T}} \log \frac{B}{S_0}$. A similar computation can be done for the case $\eta = +1$.

We summarize.

Theorem 6.5 *The supplement for one-touch digitals is given by*

$$\begin{aligned} & h(t,x;\alpha) \qquad (189) \\ = \; & Re^{(-r_d + \frac{1}{2}\alpha^2\sigma^2)\tau - \eta\alpha\sigma\sqrt{\tau}\theta_-} \\ & \left\{e^{\eta\alpha\sigma\sqrt{\tau}m} \mathcal{N}(\eta d_-) - e^{-\eta\alpha\sigma\sqrt{\tau}m + 2m\theta_-} \mathcal{N}(\eta d_+)\right\}, \\ m \; = \; & \frac{1}{\sigma\sqrt{\tau}} \ln\left(\frac{B}{x}\right) \; \textit{and } d_{\pm} = \pm m - (\theta_- - \eta\alpha\sigma\sqrt{\tau}). \end{aligned}$$

Let us note that for $\alpha = 0$ this formula simplifies to the rebate portion of a knock-in barrier option as presented, e.g., in [24].

6.2.9 Numerical Solutions

6.2.10 Range Binaries

For many option payoffs it is difficult or impossible to compute constrained value functions analytically. If the boundary conditions are known, we can still compute the constrained value function using a finite difference grid. As an example we present the commonly traded range binary option whose payoff is

$$I_{\{\min_{0\le t\le T} S_t > L; \max_{0\le t\le T} S_t < U\}} \tag{190}$$

for some lower barrier $L > 0$ and upper barrier $U > L$. If we impose the leverage constraint $\pi(t) \in [-\alpha_U, \alpha_L]$ for a given pair of nonnegative numbers $\vec{\alpha} \triangleq (\alpha_U, \alpha_L)$, then we must solve the partial differential equation defined by

$$\begin{aligned} \mathcal{L}v &= 0 \ \forall t \in [0,T), x \in (L,U), & (191)\\ v(t,x;\vec{\alpha}) &= 0 \ \forall t \in [0,T], x \notin [L,U], & (192)\\ \alpha_L v(t,L;\vec{\alpha}) - L v_x(t,L;\vec{\alpha}) &= 0 \ \forall t \in [0,T], & (193)\\ \alpha_U v(t,U;\vec{\alpha}) + U v_x(t,U;\vec{\alpha}) &= 0 \ \forall t \in [0,T], & (194)\\ v(T,x;\vec{\alpha}) &= 1 \ \forall x \in (L,U). & (195) \end{aligned}$$

make this problem homogeneous by the change of variables $y = \ln x$. The function $u(t,y) \triangleq v(t,x;\vec{\alpha})$ is then uniquely determined by

$$\begin{aligned} -r_d u + u_t + \mu u_y + \frac{1}{2}\sigma^2 u_{yy} &= 0 \ \forall t \in [0,T), y \in (\ln L, \ln U), & (196)\\ u(t,y) &= 0 \ \forall t \in [0,T], y \notin [\ln L, \ln U], & (197)\\ \alpha_L u(t,\ln L) - u_y(t,\ln L) &= 0 \ \forall t \in [0,T], & (198)\\ \alpha_U u(t,\ln U) + u_y(t,\ln U) &= 0 \ \forall t \in [0,T], & (199)\\ u(T,y) &= 1 \ \forall y \in (\ln L, \ln U), & (200) \end{aligned}$$

where we abbreviate $\mu \triangleq r_d - r_f - \frac{1}{2}\sigma^2$.

discretize the rectangle $[\ln L, \ln U] \times [0,T]$ into a uniformly spaced mesh with $M+2$ nodes along the t axis and $N+2$ nodes along the y axis:

$$\begin{aligned} y_i &= y_0 + i\Delta_y = \ln L + i\frac{\ln U - \ln L}{N+1}, \quad i = 0,\ldots,N+1, & (201)\\ t_j &= j\Delta_t = j\frac{T}{M+1}, \quad j = 0,\ldots,M+1. & (202) \end{aligned}$$

This way the boundary conditions can be captured exactly, but the initial stock value is most likely not a point in the mesh. To find the time zero value of the range binary option we must interpolate between the two neighboring mesh points of the initial stock price.

difference approximations of the partial derivatives of u. We abbreviate $u_{i,j} \triangleq u(y_i, t_j)$ and approximate

$$\begin{aligned} u_t(y_i,t_j) &\approx \frac{u_{i,j+1}-u_{i,j}}{\Delta_t}, && (203)\\ u_y(y_i,t_j) &\approx (1-\Theta)\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta_y} \\ &\quad +\Theta\frac{u_{i+1,j+1}-u_{i-1,j+1}}{2\Delta_y}, && (204)\\ u_{yy}(y_i,t_j) &\approx (1-\Theta)\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta_y^2} \\ &\quad +\Theta\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{\Delta_y^2}, && (205) \end{aligned}$$

where the parameter $\Theta \in [0,1]$ denotes the degree of explicitness.

Common values are

- $\Theta = 1$ for the fully explicit finite-difference method,
- $\Theta = 0$ for the fully implicit finite-difference method,
- $\Theta = \frac{1}{2}$ for the Crank-Nicholson scheme.

finite difference approximations of (196) yield for each $j = 0, \dots M$ the N linear equations

$$\begin{aligned} & u_{i-1,j}(-\frac{1}{2}a(1-\Theta)(\sigma^2-\Delta_y\mu)) \\ +u_{i,j}(1+r_d\Delta_t+a(1-\Theta)\sigma^2) + {} & u_{i+1,j}(-\frac{1}{2}a(1-\Theta)(\sigma^2+\Delta_y\mu)) \\ &= u_{i-1,j+1}(\frac{1}{2}a\Theta(\sigma^2-\Delta_y\mu)) \\ +u_{i,j+1}(1-a\Theta\sigma^2) + {} & u_{i+1,j+1}(\frac{1}{2}a\Theta(\sigma^2+\Delta_y\mu)), \\ & i = 1, \dots N. \end{aligned} \qquad (206)$$

boundary conditions translate into two more equations

$$\begin{aligned} & (\Delta_y\alpha_L+(1-\Theta))u_{0,j}-(1-\Theta)u_{1,j} \\ = {} & -\Theta(u_{1,j+1}-u_{0,j+1}), && (207)\\ & (\Delta_y\alpha_U+(1-\Theta))u_{N+1,j}-(1-\Theta)u_{N,j} \\ = {} & -\Theta(u_{N+1,j+1}-u_{N,j+1}). && (208) \end{aligned}$$

We obtain for each j a tridiagonal system of $N+2$ linear equations in the unknowns $u_{i,j}, i = 0, \dots, N+1$, which can be solved efficiently using an algorithm, e.g., from [22].

6.2.11 Examples

In this section, we give examples of options whose upper hedging prices can be computed using either Theorem 6.3 or 6.2. In both these theorems, the path-dependent payoff function g is assumed to be lower semicontinuous. Some option contracts are written with upper-semicontinuous payoffs. However, one can usually trivially modify an upper-semicontinuous payoff to obtain a lower-semicontinuous payoff, and then our theorems apply. Our first example highlights the danger of applying them naively to upper-semicontinuous payoffs.

Example 6.6 (Cactus option) Consider an option whose payoff at expiration date T is 1 if and only if $S(T) = K$, where K is a fixed positive number. Otherwise, the payoff is zero. The payoff can be written as $\varphi(S(T))$, where $\varphi(x) \triangleq I_{\{x=K\}}$ is upper semicontinuous rather than lower semicontinuous. If we ignore this fact and attempt to use Theorem 6.2 to compute the upper hedging price, we would first determine

$$\widehat{\varphi}_\alpha(x) \triangleq \sup_{\lambda \geq 0} e^{-\alpha\lambda}\varphi\big(xe^{-\lambda}\big) = \left(\frac{K}{x}\right)^\alpha I_{\{x \geq K\}}, \quad x \geq 0,$$

and then compute $\mathbb{E}\big[e^{-r_d T}\widehat{\varphi}_\alpha(S(T))\big]$. This last quantity is strictly positive. However, the option is clearly worth zero, since there is zero probability that $S(T) = K$. To correctly compute the upper hedging price, one should replace the given φ by its lower-semicontinuous envelope $\varphi_* \equiv 0$.

Example 6.7 (Discrete barrier option) The in-the-money knock-out call was discussed in considerable detail in Section 6.2.1. Here we modify the payoff by assuming the option can only knock out at discrete check times $0 < t_1 < t_2 < \cdots < t_I \leq T$, i.e.,

$$g(S(\cdot)) = \big(S(T) - K\big)^+ \prod_{i=1}^{I} I_{\{S(t_i) < B\}}.$$

The controlled version of the payoff function g is of the form

$$g_*(S, \lambda) = \big(S(T)e^{-\lambda(T)} - K\big)^+ \prod_{i=1}^{I} I_{\{S(t_i)e^{-\lambda(t_i)} < B\}}.$$

The supremum over all controls is approached by processes λ which are constant between the check times, and jump at the check times “just enough” to prevent knock-out. More precisely, let $\{B_n\}_{n=1}^\infty$ be converging up to B. For each n, define

$$\lambda_n(t) \triangleq \max_{\{i;\, t_i \leq t\}} \big(\log S(t_i) - \log B_n\big)^+, \quad 0 \leq t \leq T. \tag{209}$$

Then $S(t_i)e^{-\lambda_n(t_i)} \leq B_n$ for each $i \in \{1, \dots, I\}$, and λ_n is the smallest process in $\mathcal{R}$ which forces these inequalities. We obtain for the upper hedging price

$$\begin{aligned} v(0, S(0); \alpha) &= \lim_{n\to\infty} \mathbb{E}\big[e^{-r_d T - \alpha\lambda_n(T)}\big(S(T)e^{-\lambda_n(T)} - K\big)^+\big] \\ &= \mathbb{E}\big[e^{-r_d T - \alpha\lambda^*(T)}\big(S(T)e^{-\lambda^*(T)} - K\big)^+\big], \end{aligned}$$

where λ^* is given by (209) with B in place of B_n. This may be rewritten as

$$\begin{aligned} & v(0, S(0); \alpha) \\ = \quad & e^{-r_d T}\, \mathbb{E}\left[\left(1 \wedge \min_{1\leq i\leq I} \frac{B}{S(t_i)}\right)^\alpha \left(S(T)\left(1 \wedge \min_{1\leq i\leq I} \frac{B}{S(t_i)}\right) - K\right)^+\right]. \end{aligned}$$

The computation has been reduced to a finite-dimensional Gaussian integration. If the barrier depends on time, we need only to replace the ratios $B/S(t_i)$ by $B(t_i)/S(t_i)$ in the last formula.

Example 6.8 (Vanilla put) The payoff of the vanilla put is $g(S(\cdot)) = \varphi(S(T))$, where $\varphi(x) = (K - x)^+$ and K is a positive constant. According to and Theorem 6.2, the upper hedging price is

$$v(0, S(0); \alpha) \quad = \quad \sup_{\lambda\in\mathcal{R}} \mathbb{E}\big[e^{-r_d T - \alpha\lambda(T)}\big(K - S(T)e^{-\lambda(T)}\big)^+\big], \tag{210}$$

where we take $I = 1$ and $t_1 = T$ in the definition of $\mathcal{R}$, meaning that the processes are continuous except for a possible jump at time T. Theorem 6.2 applies, and asserts that $v(0, S(0); \alpha) = e^{-r_d T}\,\mathbb{E}\big[\widehat{\varphi}_\alpha(S(T); K)\big]$, where the face-lift is given by

$$\begin{aligned} \widehat{\varphi}_\alpha(x; K) &\triangleq \sup_{\lambda \geq 0} e^{-\alpha\lambda}\big(K - xe^{-\lambda}\big)^+ \\ &= \begin{cases} K - x & \text{if } 0 \leq x \leq \frac{\alpha K}{1+\alpha}, \\ \frac{K}{1+\alpha}\big(\frac{\alpha K}{(1+\alpha)x}\big)^\alpha & \text{if } x \geq \frac{\alpha K}{1+\alpha}. \end{cases} \end{aligned} \tag{211}$$

On the other hand, in the case $\alpha > 0$, maximizing the integrand in (210) for every value of $S(T)$ shows that a process $\lambda \in \mathcal{R}$ is a maximizer if

$$\lambda(T) = \left(\log S(T) - \log \frac{\alpha K}{1+\alpha}\right)^+.$$

6.2.12 Interpretation as transaction cost

delta-hedging strategy to hedge a short position, the trader will hold $v_x(t, x)$ shares of stock at time t if the stock price is x.

upon knock-out left with a position $v_x(t, B)$ in the stock valued at $|Bv_x(t, B)|$.

covering the short position requires $-\left(1 + \frac{1}{\alpha}\right) Bv_x(t, B)$ (Suppose $v_x(t, B)$ is negative)

hedging portfolio value is $v(t, B)$

wealth invested in stock is $Bv_x(t, B)$

wealth invested in the money market is $v(t, B) - Bv_x(t, B)$.

The money market position is exactly what is needed to cover the short stock position, taking the transaction cost into account, if and only if the equation $v(t, B) - Bv_x(t, B) = -\big(1 + \frac{1}{\alpha}\big)Bv_x(t, B)$ holds. This is equivalent to $\alpha v(t, B) + Bv_x(t, B) = 0$ for $0 \leq t < T$, which is condition (129) satisfied by $v^*(t, x; \alpha)$.

6.2.13 Interpretation as moving the barrier

A common practical method for dealing with up-and-out call options which knock out in the money is to price and hedge the option as if the barrier were at some level B' strictly greater than the contractual barrier B. The resulting pricing function is continuous on $[0, T] \times (0, B']$, satisfies the Black-Scholes partial differential equation on $[0, T) \times (0, B']$, is zero at the barrier B', and agrees with the call payoff $(x-K)^+$ at the expiration time T. Our function $v^*(t, x; \alpha)$ is strictly positive at $x = B$. For $\alpha > 0$ we may extrapolate it linearly above this point so that it is continuously differentiable by the formula

$$v^*(t, B; \alpha) + (x - B)\,v_x^*(t, B; \alpha), \quad x \geq B. \tag{212}$$

Because of (129), this linear extrapolation takes the value zero at $x = \left(1 + \frac{1}{\alpha}\right) B$, independently of t. Consequently, $v^*(t, x; \alpha)$ may be regarded as an approximation to the option price obtained by moving the barrier to $B' = \left(1 + \frac{1}{\alpha}\right) B$. See Figure 15.

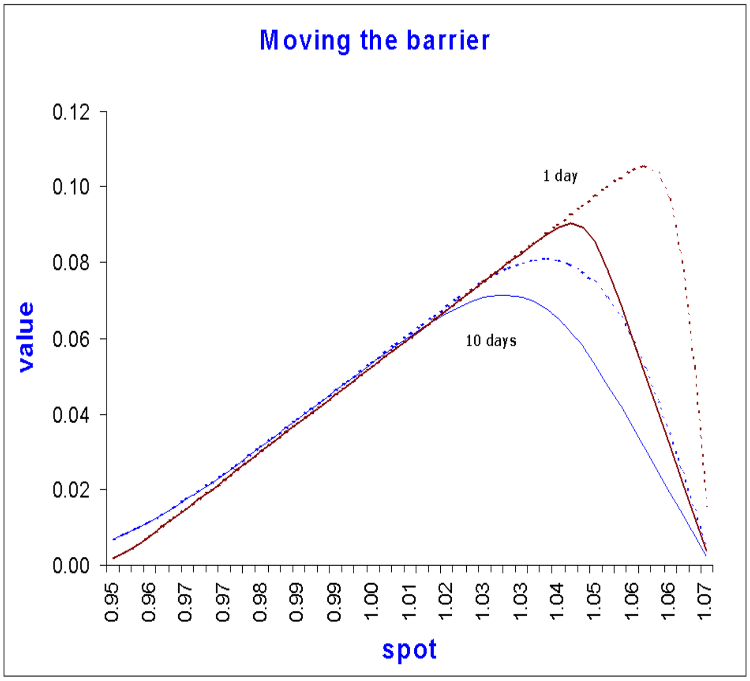


Figure 15: Upper hedging prices $v^*(0, S(0), \alpha)$ of the in-the-money knock-out calls from Figure 12. Prices are extrapolated linearly and continuously differentiable beyond the barrier $B = 1.05$ using (212). Note that the delta at the barrier is bounded below by $-\alpha(B-K)/B$. The dashed curves show the prices calculated via (109) without portfolio constraint (111) but a barrier moved to $B' = B(1 + 1/\alpha) = 1.071$. For applications, only the prices for $S(0) < B = 1.05$ are relevant.

References

[1] Andersen, L., Andreasen, J. and Eliezer, D. (2000) *Static replication of barrier options: some general results*, Working paper, General Re Financial Products.

[2] Borodin, A. N. and Salminen, P., *Handbook of Brownian Motion – Facts and Formulae*, Birkhäuser, Basel, 1996.

[3] Bouchard, B. and Touzi, N. (1999) Explicit solution of the multivariate super-replication problem under transaction costs, preprint, Université Paris Dauphine.

[4] Bowie, J. and Carr, P., Static Simplicity, *Risk,* August 1994.

[5] Broadie, M., Cvitanić, J. and Soner, H. M. (1998) On the cost of super-replication with transaction costs, *Rev. Financial Studies* **11**, 59–79.

[6] Chesney, M., Jeanblanc-Picqué, M. and Yor, M. (1997). Brownian Excursions and Parisian Barrier Options. *Adv. Appl. Prob.* **29** 165-184.

[7] Chesney, M., Cornwall, J., Jeanblanc-Picqué, M.,
Kentwell, G. and Yor, M. (1997). Parisian Pricing. *Risk* **10** ,No. 1 77-79.

[8] Cvitanić, J. and Karatzas, I. (1993) Hedging contingent claims with constrained portfolios. *Annals Appl. Probability* **3**, 652–681.

[9] Cvitanić, J., Pham, H. and Touzi, N. (1999) Super-replication in stochastic volatility models under portfolio constraints, *J. Applied Probab.* **36**.

[10] Cvitanić, J., Pham, H. and Touzi, N. (1999) A closed-form solution for the problem of super-replication under transaction costs, *Finance and Stochastics* **3**, 35–54.

[11] Derman, E., Ergener, D. and Kani, I., Forever Hedged, *Risk,* September 1994.

[12] El Karoui, N. and Quenez, M. C. (1995) Dynamic programming and the pricing of contingent claims in an incomplete market, *SIAM J. Control Optim.* **33**, 29–66.

[13] Föllmer, H. and Leukert, P. (1999) Quantile Hedging, *Finance and Stochastics* **3**, 251-273.

[14] Hakala, J. and Wystup, U. (2002) *Foreign Exchange Risk*, Risk Publications, London.

[15] Hart, I. and Ross, M. (1994). Striking Continuity. *Risk.* **7**, No. 6 51-55.

[16] Karatzas, K. and Kou, S. (1996) On the pricing of contingent claims under constraints, *Ann. Appl. Probab.* **6**, 321–369.

[17] Karatzas, K. and Kou, S. (1998) Hedging American contingent claims with constrained portfolios, *Finance and Stochastics* **2**, 215–258.

[18] Karatzas, I. and Shreve, S., *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York, 1991.

[19] Karatzas, I. and Shreve, S., *Methods of Mathematical Finance*, Springer-Verlag, New York, 1998.

[20] Linetzky, V. (1997). *Step Options (The Feynman-Kac Approach to Occupation Time Derivatives).* Technical Report 96-18 of the Department of Industrial and Operations Engineering, University of Michigan.

[21] Pechtl, A. (1998). *Some Applications of Occupation Times of Brownian Motion with Drift in Mathematical Finance.* Working Paper, DG-Bank Frankfurt.

[22] Press, W. H., Teukolksy, S. A., Vetterling, W. T. and Flannery, B. P. (1992). *Numerical Recipes in FORTRAN, Second Edition.* Cambridge University Press

[23] Revuz, D. and Yor, M. (1995). *Continuous Martingales and Brownian Motion.* Second Edition. Springer.

[24] Rich, D. (1994). The Mathematical Foundations of Barrier Option Pricing Theory. *Advances in Futures and Options Research* **7**

[25] Reiner, E. and Rubinstein, M. (1991). Breaking down the Barriers. *Risk* 4(8), 28-35.

[26] Shreve, S.E. (1996). *Stochastic Calculus and Finance.* Lecture notes, Carnegie Mellon University

[27] Schmock, U., Shreve, S.E. and Wystup, U. (2000) Valuation of exotic options under shortselling constraints, Working paper, Swiss Federal Institute of Technology.

[28] Soner, H. M. and Touzi, N. (1999) Super-replication under gamma constraint, preprint, Dept. Operations Research and Financial Engineering, Princeton University.

[29] Touzi, N. (1999) Direct characterization of the value of super-replication under stochastic volatility and portfolio constraints, preprint, Université Paris Dauphine.

[30] Wystup, U (2000). The MathFinance Formula Catalogue. `http://www.mathfinance.de`

[31] Zhang, P., *Exotic Options: A Guide to Second Generation Options*, World Scientific, Singapore, 1997.